Left-Degenerate Vacuum Metrics

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For all complex space-times in which the self-dual part of the Weyl tensor is algebra-}
ically degenerate, Einstein's vacuum equations are reduced to a single differential equa-
tion of the second order and second degree.

It is well known that Einstein's vacuum equa-
tions can be simplified considerably if the space-
time admits a congruence of null shear-free geo-
desics, or if the Weyl tensor is anti-self-dual. Here we shall consider a broad class of complex metrics which includes both these as special cases.

We impose only one restriction on our space-
time: that it admits a congruence of totally null surfaces. To describe them, we introduce the surface element

$$\Sigma_{ab} = u^a v_b - v^a u_b ,$$

and the expansion form

$$\theta = \theta_a dx^a = \frac{1}{2} (u^a_v dv - v^a_u du) ,$$

where $u$ and $v$ are functionally independent scalars, constant on each surface. By a totally null surface we mean a differentiable two-space to which all tangent vectors are null. It follows that $du$ and $dv$ are null and mutually orthogonal. From this, one can easily prove that

$$\Sigma_{ab} \Sigma^{re} + \Sigma_{eb} \theta^e = 0 .$$

In the special case $\theta = 0$, not merely is $\Sigma$ covari-
antly constant on each surface, but the equations

$$X^e_r \Sigma^{rb} = 0$$

have a tetrad of independent solutions. A totally null surface, therefore, is geodesic by definition, and plane if its expansion vector is zero.

The surface element is self-dual or anti-self-
dual. We describe the congruence as left-handed in the first case, and right-handed in the second. A congruence of null shear-free geodesics is the intersection of a left-handed congruence of totally null surfaces with a right-handed one. Here, of course, we are dealing with only one congruence. We take it to be left-handed.

For our purposes, the empty-space equations fall naturally into three classes: first, the three surface equations,

$$\Sigma_a^b R_{pa} \Sigma^{eb} = 0 ;$$

second, the central equations, comprising $R = 0$ and the three remaining equations of

$$R_{ab} \Sigma^{eb} = 0 ;$$

and third, the three residual equations of $R_{ab} = 0$. Since $\Sigma$ is self-dual, null, and closed, the equa-
We use them to put $\varphi = ay - bx + c$, with constant $a$, $b$, and $c$. The expansion form is now given by

$$\theta = \varphi (a \, du + b \, dv).$$

The metric belongs to the Plebanski-Schild class,

$$ds^2 = ds_0^2 + 2\varphi^{-2}(\sigma \, du + 2\beta \, dv),$$

where $ds_0^2$ is flat, while $du$ and $dv$ are null and orthogonal. In the special case

$$\Delta = \varphi^{-2}(\sigma \, q - 6\beta^2) = 0,$$

it reduces to the Kerr-Schild form.6

The central equations are more complicated. After some manipulation, one finds that the general solution contains three disposable functions: $\Pi(u, v, x, y)$, $f(u, v)$, and $g(u, v)$. It may be written as

$$\varphi^{-3}/\theta = \xi f - (\varphi^{-2}\Pi)_y,$$

$$\varphi^{-3}/\eta = \eta g - (\varphi^{-2}\Pi)_x,$$

$$2\varphi^{-3}/\theta = \xi f + (\varphi^{-2}\Pi)_y + (\varphi^{-2}\Pi)_x,$$

where $\xi = f$ and $\eta = g$ for $\theta \neq 0$, while $\xi = \xi x$ and $\eta = \xi y$ for $\theta = 0$. One can derive the second case as a limit of the first. The residual equations take the form

$$E_{xx} = E_{xy} = E_{yy} = 0,$$

with the integral

$$2E = y\alpha(u, v) - x\beta(u, v) + \gamma(u, v),$$

where $E$ is constructed from $a$, $b$, $c$, $f$, $g$, and $\Pi$. This is our one remaining field equation.

In the diverging case, we obtain $a = -b = 1$, $c = 0$, $f = g = \frac{1}{2}\sqrt{\mu}$, by specializing the coordinates, and using the transformation

$$f \rightarrow f + 2ak, \quad g \rightarrow g + 2bk,$$

$$\Pi \rightarrow \Pi + k\varphi^3(fy - gx + k\varphi),$$

where $k$ is a disposable function of $u$ and $v$. We then find that

$$E_\Delta = \Delta + \varphi^{-2}(\Pi_x - \Pi_y)^2 + \frac{1}{2}\mu\varphi(\Pi_x + \Pi_y) - 3\mu\Pi + \varphi^{-1}(\Pi_{xv} + \Pi_{yy}) - \frac{1}{4}(x - y)(x\mu_u - y\mu_v).$$

We make $\mu$ constant and put $\alpha = \beta$ by specializing the coordinates further and using the transformation

$$\Pi \rightarrow \Pi + \frac{1}{2}\varphi^{-2}(\alpha, \beta, u, v), \quad \alpha = \alpha + t_u + l_v, \quad \beta = \beta - t_u - l_v.$$

The self-dual components of the Weyl tensor are given by7

$C^{(5)} = C^{(4)} = 0$, $C^{(3)} = -2\mu\varphi^3$, $C^{(2)} = 2\beta\varphi_5$,

$$C^{(1)} = 2\varphi^3\left[ y_\beta_\alpha - x_\beta_\alpha - 2\beta(\Pi_x - \Pi_y) + (\theta_u + \theta_v)\left\{ \frac{1}{2} y - \mu\varphi^{-1/2}\theta_x + \theta_y \right\} \varphi^{-3/2}\Pi \right];$$

the anti-self-dual components, by

$$C^{(m)} = 2\varphi^3\delta_\alpha^n \delta_\beta^m \Pi, \quad n = 1, \ldots, 5.$$
In the plane case, it is convenient to put $\phi = 1$. We then have

$$\Xi = \Delta + \delta_u \Pi_x + \delta_v \Pi_y + \frac{1}{2}(y\delta_u - x\delta_v)(yf - xg),$$

where

$$\delta_u \equiv \delta_u + f, \quad \delta_v \equiv \delta_v + g.$$  

We can make $\alpha = \beta = \gamma = 0$ by means of a transformation on $\Pi$; but this has the effect of introducing into the metric additional functions $\rho$, $q$, and $r$ of $u$ and $v$:

$$\phi' = -\Pi_{yy} + p + \frac{3}{8}xf, \quad \phi = -\Pi_{xx} + q + \frac{3}{8}yg, \quad \phi' = \Pi_{xy} + r + \frac{1}{8}(yf + xg).$$

When the field equation is satisfied, the Weyl tensor is given by

$$C^{(5)} = C^{(4)} = C^{(3)} = 0, \quad C^{(2)} = f_y - g_u, \quad C^{(1)} = (y\delta_u - x\delta_v)C^{(2)} - 2\delta_u^2\rho - 2\delta_u^2q + 4\delta_u\delta_vr,$$

and

$$\bar{C}^{(n)} = 2\delta_u\delta_v\delta_u\delta_v^{n-1}II, \quad n = 1, \ldots, 5.$$  

Without loss of generality, we can make one function zero in each of the sets $\{f, g\}$ and $\{\rho, q, r\}$; if the Weyl tensor is left-null, we can make $f = g = 0$; if left-flat, $f = g = \rho = q = r = 0$.

One can deal with Einstein-Maxwell vacuum equations in much the same way, provided that one takes

$$\Sigma_{ab}F^{ab} = 0.$$  

The surface equations are unaltered; Maxwell's equations give

$$F_{ab}dx^a \wedge dx^b = \epsilon(du \wedge dx + dv \wedge dy) + (\delta + x\epsilon_u - y\epsilon_v)du \wedge dv$$

$$+ \phi[H_{xx}e^2 \Lambda e^3 + H_{xy}e^1 \Lambda e^4 + H_{yy}(e^1 \Lambda e^2 + e^2 \Lambda e^3)],$$

where the wedges indicate antisymmetrical tensor multiplication, $\epsilon$ and $\delta$ are disposable functions of $u$ and $v$ only, while $H$ is subject to

$$H_{uu} + H_{vv} = \delta H_{xx} + 2 \delta H_{yy} + 2 \delta H_{xy};$$

and the remaining equations integrate in much the same way as in the purely gravitational system.  

There is an interesting formal resemblance between the roles of $\Pi$ in the last metric and $H$ here.

The present work might well simplify the problem of finding real degenerate solutions in the case that has so far proved most refractory: that of twisting rays. Of greater interest, however, is the possibility of moving in the opposite direction: not specializing the anti-self-dual part of the Weyl tensor, but removing the present restriction on its self-dual part. This would presumably involve the introduction of a second Hertz function $\tilde{H}$. Our conjecture is that Einstein's equations in the most general complex case could be reduced to a pair of differential equations of the second order and second degree.

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7. The expression for $C^{(1)}$ given here was derived by J. D. Finley, III, and A. Garcia.

8. Details of this generalization will be given in a paper by A. Garcia and the present authors.