Heavens and their integral manifolds

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By using the apparatus of exterior forms, a new spinorial notation and Cartan's theory of integral manifolds, some new results concerning complex strong heavenly metrics are established. In particular, the study of a subfamily of two-variable heavens (types $G\Theta[\cdot]$, $D\Theta[\cdot]$, and $N\Theta[\cdot]$) is reduced to linear equations, a prolongation process related to first and second heavenly equations is established leading to a (presumably) infinite hierarchy of 1-forms, and finally, the symmetries of the studied structure are investigated from the point of view of its description by Pfaffian forms, elucidating in this way previous results concerning Killing vectors.

1. INTRODUCTION

This paper is the fifth in a series of articles dedicated to the study of the analytic continuation of general relativity, with special emphasis on the solutions of the complex Einstein equations characterized by the self-dual conformal curvature. These spaces have been called heavens by Newman and Penrose. The first article of the series outlined the formalism of complex tetrads, forms, and spinors used subsequently and established heavens as the integral manifolds of a certain partial differential equation of order and degree 2. Actually, two equivalent partial differential equations for a single function were given—the first and second heavenly equations. In the second article, a generalization of the Goldberg–Sachs theorem to complex Riemannian spaces was given, elucidating the important role of complex null strings. Then in many explicit heavens of various algebraic type were studied and the problem of finding the conformal Killing vectors for an arbitrary heaven was reduced to a single equation while the general theory of Killing spinors in both real and complex Riemannian spaces was studied in Ref. 6.

The purpose of this article is to study the general integral manifolds of heaven from a geometric point of view. While the general form of a regular integral manifold is only given implicitly, many of its properties are obtained and studied form the point of view of both a concise spinorial language and closed Pfaffian 1-forms. The outline of the paper is as follows: in order to make the succeeding sections more palatable we show first in Sec. 2 how a special subcase of two-variable heavens can be solved completely by the systematic use of elementary exterior differential calculus which reduces the problem to the linear two-dimensional complex Laplace equation. For this case we give a complete classification of the self-dual conformal curvature types.

In Sec. 3 we organize some of the basic results of the previous papers in the concise spinorial language, while Sec. 4 treats the integral manifolds in terms of pairs of Pfaffian 1-forms. The idea of prolongations first introduced by Cartan is used to study further properties, in particular the relation between the first and second heavenly equations and establishing a hierarchy (presumably infinite) of 1-forms. Then in Sec. 5 we investigate the structure of the regular integral manifolds of heavens from the point of view of the general Cartan theory. Furthermore, some explicit results concerning subcases when the problem can be reduced to linear structures are presented. Finally, in the last section the symmetry group which maps the heavenly integral manifolds into each other is computed and the relation to Killing vectors is discussed.

2. TWO-VARIABLE HEAVENS

In Ref. 5 the general solution of the reduced two variable problem [Eq. (2.1) below] was solved using the method of first integrals. However, since the computations involved were quite complicated, the classification of the algebraic degeneracy of the conformal curvature was not given. In this section we present this classification as well as the general form of the metric, connections, and curvature using fairly simple techniques illustrate the ease in which differential forms can be used to solve concrete problems which at first sight appear formidable. The reduced two variable equation of Ref. 5 is

\[ \Theta_{x_{\alpha}}\Theta_{y_{\beta}} - \Theta_{x_{\beta}}\Theta_{y_{\alpha}} = 1. \]  

(2.1)

Here we have transformed the constant $-k^2$ in Eq. (2.34) of Ref. 5 to 1 by a complex dilatation. The case when $k^2 = 0$ was completely solved in Ref. 5 and yields algebraically special metrics. We will also briefly discuss this case in the present context. In both cases we succeed in linearizing the theory.

To write (2.1) in differential form language we first write the contact 1-form

\[ d\Theta - u\, dx - v\, dy = 0, \]  

(2.2)

which implies $u = \Theta_x$, $v = \Theta_y$. Then it is easy to see that (2.1) becomes

\[ du \wedge dv - dx \wedge dy = 0. \]  

(2.3a)

Taking the exterior derivative of (2.2) we find

\[ du \wedge dx + dv \wedge dy = 0. \]  

(2.3b)

Now we can consider (2.1) to be equivalent to (2.3) with
After a little straightforward algebra we obtain the components of the conformal curvature,

\[ C^{(3)} = \frac{1}{4} (F + \bar{F})^4 \left( \frac{\bar{F}^2 + F^2}{F + \bar{F}} \right), \]

\[ C^{(5)} = 2i (F + \bar{F})^4 \left( \frac{F^2 - \bar{F}^2}{F + \bar{F}} \right) \left( \frac{F^2 + \bar{F}^2}{F + \bar{F}} \right). \]
3. STRONG HEAVENS IN THE SPINORIAL NOTATION

The results derived in Ref. 3 concerning strong heavens were obtained by working with the spinorial formalism; however they were stated in a notation which did not make explicit use of the spinorial character of the various quantities concerned. We can now considerably improve the condensed presentation of these results by the simple device of introducing in place of the variables \( \{xypq\} \) and \( \{pqrs\} \) which were used in Ref. 3, the new variables defined by

\[
x := -p^1, \quad y := -p^2, \quad p := q_1, \\
q := q_2, \quad r := \bar{q}_1, \quad s := \bar{q}_2
\]

which will be interpreted as formal spinors \( (p^A, q_A, \bar{q}_A) \). The spinorial indices should be then manipulated according to the standard rules

\[
\psi^A := \epsilon_{AB} \psi^B, \quad \bar{\psi}^A := \epsilon^{AB} \bar{\psi}_B, \\
\bar{\psi}^A := \epsilon^{AB} \bar{\psi}_B, \quad \bar{\psi}^A := \epsilon_{AB} \psi^B.
\]

The key functions depend now on their respective variables written in spinor notation, viz.

\[
\Omega := \Omega(q_A, \bar{q}_A), \quad \Theta := \Theta(p^A, q_A).
\]

The corresponding heavenly equations assume the form

\[
\begin{align*}
\frac{1}{2} \frac{\partial^2 \Omega}{\partial q_A \partial \bar{q}_B} + \frac{1}{2} \frac{\partial^2 \Omega}{\partial \bar{q}_A \partial q_B} + 1 = 0, \\
\frac{1}{2} \frac{\partial^2 \Theta}{\partial p^A \partial \bar{p}_B} + \frac{1}{2} \frac{\partial^2 \Theta}{\partial \bar{p}_A \partial p_B} + \frac{1}{2} \frac{\partial^2 \Theta}{\partial q_A \partial q_B} = 0,
\end{align*}
\]

and the (strongly) heavenly metric takes the form of

\[
H := ds^2 = 2e^1 \otimes e^3 + 2e^3 \otimes e^1
\]

\[
= \frac{\partial^2 \Omega}{\partial q_A \partial \bar{q}_B} dq_A \otimes dq_B + \frac{1}{2} \frac{\partial^2 \Theta}{\partial p^A \partial \bar{p}_B} dp^A \otimes dp_B + \frac{1}{2} \frac{\partial^2 \Theta}{\partial q_A \partial q_B} dq_A \otimes dq_B.
\]

The heavenly tetrad and its inverse are then given by

\[
(g^A_b) = \sqrt{2} \begin{pmatrix} e^1 & -e^3 \\ e^3 & e^1 \end{pmatrix} = \sqrt{2} \begin{pmatrix} dq_A, & \frac{\partial^2 \Omega}{\partial q_A \partial \bar{q}_B} dq_B \\ dq_B, & \frac{\partial^2 \Omega}{\partial \bar{q}_A \partial q_B} dq_B \end{pmatrix}
\]

\[
= \sqrt{2} \begin{pmatrix} dq_A, & -dp^1 \frac{\partial^2 \Theta}{\partial p^B \partial q_A} \\ dq_B, & -dp^3 \frac{\partial^2 \Theta}{\partial \bar{p}_B \partial q_A} \end{pmatrix}.
\]

For the invariant d'Alembertian in the strong heavens we find

\[
\Box \Phi := \nabla_a \nabla^a \Phi = 2(a_2 \partial_1 + a_3 \partial_4) \Phi
\]

\[
= -2 \frac{\partial^2 \Omega}{\partial q_A \partial \bar{q}_B} \frac{\partial^2 \Theta}{\partial q_A \partial q_B} \Phi.
\]

Now, the basic relation which establishes the bridge between the \( \Omega \) and \( \Theta \) formalisms is

\[
p^A = -\frac{\partial \Omega}{\partial q_A}.
\]

It will be useful to consider a parallel object

\[
\bar{p}^A = -\frac{\partial \Omega}{\partial \bar{q}_A}.
\]

Then, for the base of the (closed anti-self-dual 2-forms we have

\[
\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = (q^A \bar{q}_B) = (\epsilon_{AB}).
\]

The invariant volume along \( V_4 \) is given by

\[
-\frac{\partial \Omega}{\partial q_A} \frac{\partial \Theta}{\partial q_A} dq_A \wedge dq_B \wedge dq_C \wedge dq_D.
\]

At the same time, for the base of the self-dual 2-forms we have

\[
(\epsilon_{AB}) = \begin{pmatrix} 2e^1 \wedge e^3 & -e^1 \wedge e^3 + e^3 \wedge e^1 \\ -e^1 \wedge e^3 + e^3 \wedge e^1 & 2e^3 \wedge e^1 \end{pmatrix}
\]

\[
= \begin{pmatrix} dq^A \wedge dq_B & -dq^A \wedge dq_B \\ -dq^A \wedge dq_B & dq^A \wedge dq_B \end{pmatrix}.
\]

The invariant volume along \( V_4 \) is given by

\[
-\frac{\partial \Omega}{\partial q_A} \frac{\partial \Theta}{\partial q_A} dq_A \wedge dq_B \wedge dq_C \wedge dq_D.
\]

\[
= \begin{pmatrix} 2dq^A \wedge dp^B + \frac{\partial \Theta}{\partial q_A} \frac{\partial \Omega}{\partial \bar{q}_A} dq_C \wedge dq_D \\ 2dq^A \wedge dp^B - \frac{\partial \Theta}{\partial q_A} \frac{\partial \Omega}{\partial \bar{q}_A} dq_C \wedge dq_D \end{pmatrix}.
\]

The invariant volume along \( V_4 \) is given by

\[
-\frac{\partial \Omega}{\partial q_A} \frac{\partial \Theta}{\partial q_A} dq_A \wedge dq_B \wedge dq_C \wedge dq_D.
\]

At the same time, for the base of the self-dual 2-forms we have

\[
(\epsilon_{AB}) = \begin{pmatrix} 2e^1 \wedge e^3 & -e^1 \wedge e^3 \wedge e^3 \wedge e^1 \\ e^1 \wedge e^3 \wedge e^3 \wedge e^1 & 2e^3 \wedge e^1 \end{pmatrix}
\]

\[
= \begin{pmatrix} dq^A \wedge dq_B \wedge dq_C \wedge dq_D & -dq^A \wedge dq_B \wedge dq_C \wedge dq_D \\ -dq^A \wedge dq_B \wedge dq_C \wedge dq_D & dq^A \wedge dq_B \wedge dq_C \wedge dq_D \end{pmatrix}.
\]

The invariant volume along \( V_4 \) is given by

\[
-\frac{\partial \Omega}{\partial q_A} \frac{\partial \Theta}{\partial q_A} dq_A \wedge dq_B \wedge dq_C \wedge dq_D.
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\]

\[
= \begin{pmatrix} dq^A \wedge dq_B \wedge dq_C \wedge dq_D & -dq^A \wedge dq_B \wedge dq_C \wedge dq_D \\ -dq^A \wedge dq_B \wedge dq_C \wedge dq_D & dq^A \wedge dq_B \wedge dq_C \wedge dq_D \end{pmatrix}.
\]
The curvature form
\[ R_{AB} := d\Gamma_{AB} + \Gamma_{AC} \wedge \Gamma_{CB} \]
\[ = \frac{1}{2} \frac{\partial^2 \Omega}{\partial p^A \partial p^B} \wedge \frac{\partial^2 \Omega}{\partial \nu^C \partial \nu^D} \cdot (dp^C + \frac{\partial^2 \Omega}{\partial p^A \partial \nu^C} dq^A) \]
\[ = -\frac{1}{4} C_{ABCD} \delta^{CD}. \] (3.17)
determines the only nontrivial spinorial curvature quantity
\[ C_{ABCD} = \frac{\partial^2 \Omega}{\partial p^A \partial p^B} \delta^{CD}. \] (3.18)
Of course, because H with the null tetrad oriented as in (3.7) is a strong heavy, \( C_{ABCD} = 0 = R_{AB} \). The heavenly conformal curvature can also be expressed in terms of the first key function
\[ C_{ABCD} = \frac{\partial^2 \Omega}{\partial q^A \partial q^B} \left( \frac{\partial^2 \Omega}{\partial q^A \partial \nu^C} \right) \left( \frac{\partial^2 \Omega}{\partial q^B \partial \nu^D} \right) \left( \frac{\partial^2 \Omega}{\partial q^C \partial q^D} \right). \] (3.19)
The symmetrization affects here only the undotted indices \( ABCD \).

We will mention that from the two expressions for \( ds^2 \) in (3.6) one directly infers that
\[ \frac{\partial^2 \Omega}{\partial q^A \partial q^B} + \frac{\partial^2 \Omega}{\partial \nu^A \partial \nu^B} = 0. \] (3.20)

Now, a basic advantage of the present notation is that if we restrict the heavenly factor of the gauge group \( SL(2, \mathbb{C}) \) in \( \mathcal{G} = SL(2, \mathbb{C}) \times SL(2, \mathbb{C}) \) to constant transformations, it coincides directly with the freedom of \( SL(2, \mathbb{C}) \) transformations of our formal spinors. These transformations (which maintain the simple expression for \( \Gamma_{AB} \) in the terms of \( \Omega \)) represent the ambiguity group of the present spinorial description of strong heavens in the \( \Omega \) formalism. Notice that according to (3.12) the "hellish" 2-forms \( \ast \) are invariants of this group, as it should be. It should be observed, however, that working with the \( \Omega \) formalism, we then have two independent ambiguity transformations of our formal spinors, \( SL(2, \mathbb{C}) \) and \( SL(2, \mathbb{C}) \), with constant coefficients, where \( SL(2, \mathbb{C}) \) does not coincide with the hellish factor in \( \mathcal{G} \) restricted to constant transformations. This fact should be remembered when working with the \( \Omega \) formalism if one wants to avoid confusions.

4. PROLONGATIONS AND THE DERIVATION OF THE HEAVENLY STRUCTURE

In this section we will apply Cartan's idea of prolongation\(^\dagger\) to study the heavenly integral manifolds and their relation to a hierarchy of key functions. Since the procedure of prolongation of an ideal of differential forms is probably new to the reader we will proceed rather cautiously. We will also employ the spinorial formalism developed in the preceding section. Our treatment here is local; however, the formalism used is readily applicable to a global treatment. The standard mathematical device for patching the local information together to obtain a global theory is to use the theory of algebraic sheaves.\(^2\) Then the problem of the existence of exact global 1-forms along some integral manifold is a problem involving sheaf cohomology theory. We only mention this as a future road to the global theory and all our integrations here will be in star shaped regions.

Now consider two independent 4-forms given in local coordinates by
\[ \alpha := (dp^1 \wedge dq_1 + dp^2 \wedge dq_2) \wedge dq_1 \wedge dq_2 \]
\[ = \frac{1}{2} (dp_A \wedge dq_A) \wedge (dq_A \wedge dq_A), \] (4.1a)
\[ \beta := (dp^1 \wedge dp^2 + dq_1 \wedge dq_1) \wedge dq_1 \wedge dq_2 \]
\[ = \frac{1}{2} (dp_A \wedge dp_A + dq_A \wedge dq_A) \wedge (dq_A \wedge dq_A), \] (4.1b)
involving six complex variables \( \{p_A, q_A, \bar{q}_A\} \). We can take the global manifold here as \( \mathbb{C}^6 \). Now an integral manifold \( I \) of the ideal of differential forms generated by (4.1) is a pair \( (N, i) \) where \( N \) is an analytic manifold and \( i : N \rightarrow \mathbb{C}^6 \) is an immersion (locally 1–1) such that the pullback \( i^* \omega := \omega(i(p)) = 0 \) for \( p \in N \) and \( \omega \) in the ideal. Hereafter, we will take poetic license and simply write an integral manifold \( I \) as any subspace which satisfies
\[ \alpha = 0, \quad \beta = 0. \] (4.2)
By external multiplication of \( \alpha \) and \( \beta \) by the basis 1-forms \( \{dp_A, dq_A, \bar{d}q_A\} \) one easily finds that all \( \theta = \{q\} \) of the possible external products of five differentials of these variables vanish on \( I \) as a consequence of (4.2). Therefore, for any integral manifold we have
\[ \dim I = 4. \] (4.3)
We are interested in exactly four–dimensional integral manifolds along which we explicitly assume
\[ 0 \ast dV = -\frac{1}{2} dq^A \wedge dq_A \wedge dp_A \wedge dp_B \]
\[ = \frac{1}{2} dq^A \wedge dq_A \wedge dp^A \wedge dp^B, \] (4.4)
i.e., the ideal generated by \( \alpha \) and \( \beta \) satisfying (4.2) is an involutive\(^9\) with respect to either set of variables \( \{q_A, p_A\} \) or \( \{\bar{q}_A, q_A\} \). The last equality in (4.4) follows from \( \beta = 0 \).

Now with \( dV \neq 0 \) we can select in particular, \( \{\bar{q}_A, q_A\} \) as local independent coordinates for \( I \), then having
\[ p_A = p_A(q_A, \bar{q}_A) \] (4.5)
so that Eqs. (4.1) and (4.2) become
\[ \alpha = \left( -\frac{\partial p_A}{\partial q_1} + \frac{\partial \bar{p}_A}{\partial q_1} \right) dV = 0, \] (4.6a)
\[ \beta = \left( \frac{\partial \bar{p}_A}{\partial q_2} - \frac{\partial p_A}{\partial q_2} \right) dV = 0. \] (4.6b)
As a consequence of the local inversion of the Poincaré lemma, (4.6a) implies the existence of a function
\[ \Omega = \Omega(q_A, \bar{q}_A) \] such that
\[ p_A = -\frac{\partial \Omega}{\partial q_A}. \] (4.7)
Plugging (4.7) into (4.6b) we recover the first heavenly equation (3.5a). Thus our integral manifolds \( I \) of (4.2) are, at least locally, in 1–1 correspondence with the solutions of the first heavenly equation. We will now analyze various consequences of Eq. (4.2).

In order to proceed systematically with the program, we will now state an elementary lemma\(^11\) (which will heretofore be referred to as L):

**Lemma L:** Let \( \mathcal{D} \) denote a star-shaped region of a complex analytic manifold \( M \) of complex dimension \( n \),
and let $x^1, \ldots, x^k$ be $k$ independent local coordinates in $\mathcal{M}$, i.e., $dx^1 \wedge dx^2 \wedge \cdots \wedge dx^k = 0$, $k = n$. Let $e$ be a 1-form ($e \in \Lambda^1$) such that in $\mathcal{L}^5 = dx^1 \wedge dx^2 \wedge \cdots \wedge dx^k = 0$. Then there exists in $\mathcal{L}_0$ 0-forms $x, y_1, \ldots, y_n \in \Lambda^0$ such that

$$e = dx + \sum_{i=1}^n y_i \, dx^i.$$  

(4.8)

We can now rewrite (4.1a) and (4.1b) in the form

$$\alpha = d(p^A dq_A) + dq_1 \wedge dq_2,$$  

(4.9a)

$$\beta = \frac{1}{2}(p^A dq_A + q^A dq_A) \wedge dq_1 \wedge dq_2,$$  

(4.9b)

and applying $\mathcal{L}$ to the integral manifolds for which $\alpha = \beta = 0$, i.e., (4.2) satisfied, we infer the existence of functions $\Omega, \Sigma$ and $s^4$ such that

$$p^A dq_A + q^A dq_A = -d\Omega,$$  

(4.10a)

$$s^4 dq_A + \frac{1}{2} p^A dq_A + q^A dq_A = d\Sigma.$$  

(4.10b)

Differentiating these relations we have of course

$$dp^A \wedge dq_A = -dp^2 \wedge dq_A, \quad \frac{1}{2}(dp^A \wedge dp_A + dq^A \wedge dq_A) = -ds^4 \wedge dq_A.$$  

(4.11a)

(4.11b)

Now, the new spinors which have appeared in these relations are $\tilde{P}_A$ and $\tilde{S}_A$. We can now observe that the pairs of spinors $\{P_A, \tilde{P}_A\}$ and $\{S_A, \tilde{S}_A\}$ play symmetric roles. Indeed, multiplying externally (4.11a) by $dq_1 \wedge dq_2$ we deduce the equation

$$\tilde{a} := (dp^1 \wedge dq_1 + dp^2 \wedge dq_2) \wedge dq_1 \wedge dq_2 = \frac{1}{2}(dp^1 \wedge dq_1 \wedge dq_2 \wedge dq_A = 0.$$  

(4.12)

On an integral manifold. Now take the external “squares” of both sides of (4.11a); this gives

$$dq_1 \wedge dq_2 \wedge dp^1 \wedge dp^2 = \frac{1}{2} dq^A \wedge dq_A \wedge dq_1 \wedge dq_2 \wedge dp^4 \wedge dp_A = dq_1 \wedge dq_2 \wedge dp^1 \wedge dp^2 = \frac{1}{2} dq_A \wedge dq_1 \wedge dq_2 \wedge dp^4 \wedge dp_A.$$  

(4.13)

Consequently, the functions $\{\tilde{P}_A, \tilde{S}_A\}$ are independent.

Now, eliminating $dp^A \wedge dp_B$ in equality (4.13) by the use of (4.11b) leads to

$$0 = \tilde{a} := (dp^1 \wedge dp^2 + dq_1 \wedge dq_2) \wedge dq_1 \wedge dq_2 = \frac{1}{2}(dp^1 \wedge dq_1 \wedge dp^2 \wedge dq_1 \wedge dq_2 = 0.$$  

(4.14)

Thus, equations $\alpha = \beta = 0$ and $\tilde{a} = \tilde{a} = 0$ imply each other and are related by the formal transformation $\{P_A, S_A, \tilde{P}_A, \tilde{S}_A\} \to \{P_A, S_A, \tilde{P}_A, \tilde{S}_A\}$.  

Now, $\tilde{a} = 0$ by the application of $\mathcal{L}$ and again gives us (4.1a). From $\beta = 0$, however, by the application of $\mathcal{L}$ we obtain the information that there exist functions $s^4$ and $\Sigma$ such that

$$s^4 dq_A + \frac{1}{2} p^A dq_A + q^A dq_A = d\Sigma.$$  

(4.15)

This relation differentiated gives, of course,

$$\frac{1}{2}(dp^A \wedge dp_A + dq^A \wedge dq_A) = -ds^4 \wedge dq_A.$$  

(4.16)

It is now clear that the equations $\tilde{a} = \tilde{a} = 0$ again lead, through the elimination of $\tilde{P}_A$ in the form of $\tilde{P}_A = \partial x^i / dq_i$, to the first heavenly equation, (3.5a). Therefore, we can now equivalently state the problem of the integral manifold as follows: Postulating simultaneously any of the two pairs of equations in 1-forms,

$$\begin{align*}
\mathcal{A} &= \left\{ \begin{array}{l}
\varepsilon dq_A = \frac{1}{2} p^A dq_A + s^4 dq_A + q^A dq_A,
\end{array} \right. \\
&\begin{array}{l}
\varepsilon dq_A = \frac{1}{2} p^A dq_A + s^4 dq_A + q^A dq_A.
\end{array}
\end{align*}$$  

\begin{align*}
\mathcal{A} &= \left\{ \begin{array}{l}
d\Sigma = \frac{1}{2} p^A dq_A + s^4 dq_A + q^A dq_A,
\end{array} \right. \\
&\begin{array}{l}
d\Sigma = \frac{1}{2} p^A dq_A + s^4 dq_A + q^A dq_A.
\end{array}
\end{align*}$$  

(4.17)

one is led to the first heavenly equation. Thus, it is reasonable, instead of considering separately the pair $\mathcal{A}$ or the pair $\mathcal{B}$, to consider the three equations in (4.17) as 1-forms where there enter six spinors $\{P_A, S_A, \tilde{P}_A, \tilde{S}_A; q^A, \bar{q}^A, \tilde{q}^A\}$ and the three key functions $(\Omega, \Sigma, \Sigma)$ together as equations which determine an integral manifold in the corresponding multidimensional space.

We shall thus call the three relations in (4.17) the nucleus $\mathcal{N}$ of the heavenly structure of 1-forms. The integrability conditions of $\mathcal{N}$ have of course the shape of three equations in 2-forms,

$$\begin{align*}
\frac{1}{2}(dp^A \wedge dq_A + s^4 \wedge dq_A + q^A \wedge dq_A) &= 0, \\
\frac{1}{2}(dp^A \wedge dq_A + s^4 \wedge dq_A + q^A \wedge dq_A) &= 0.
\end{align*}$$  

(4.18)

(4.19)

(Of course, it is enough to postulate $\partial \mathcal{A}$ in order to deduce $\mathcal{A}$ and vice versa.)

Now, we are going to show that $\mathcal{N}$ forms a natural part of some much wider structure of 1-forms, which, among other things, also describes the integral manifold of the second heavenly equation. For this purpose, we first respectively eliminate in the expressions for $\alpha$ and $\tilde{a}$ [the formulas (4.1) and (4.12)] $dq_A \wedge dq_A \wedge dq_A$ by using (4.11b), and $dq_A \wedge dq_A$ by using (4.16); this leads to the equations

$$\begin{align*}
\sigma^{s1} := (dp_A \wedge dq_A) \wedge (ds_A \wedge dq_A) &= 0, \\
\tilde{\sigma}^{s1} := (dp_A \wedge dq_A) \wedge (d\bar{q} \wedge dq_A) &= 0.
\end{align*}$$  

(4.19a)

(4.19b)

At this point it is convenient to observe that the numerical identity

$$\xi = d \xi = 0$$  

(4.20)

(any object skew in the three indices in two dimensions vanishes), when contracted with $dk^4 \wedge dl^B \wedge dm^C \wedge dm$ provides a general $\Lambda^1$ identity

$$G : dk^4 \wedge dl^A \wedge dm^B \wedge dm_A \wedge dl^B \wedge dm_B + dk^4 \wedge dl^A \wedge dm^B \wedge dm_B = 0.$$  

(4.21)

In particular, identifying here $m^4 = n^4$ we obtain a special identity

$$S : dk^4 \wedge dl^A \wedge dm_B - \frac{1}{2} dk^4 \wedge dl^A \wedge dm_B.$$  

(4.22)

Now, by using $S$, we can rewrite (4.19a)–(4.19b) in the form

$$\begin{align*}
\alpha^{s1} &= -\frac{1}{2} dq_A \wedge dp_A \wedge dq_A + q^A \wedge dq_A = 0, \\
\tilde{\alpha}^{s1} &= -\frac{1}{2} dq_A \wedge dp_A \wedge dq_A + q^A \wedge dq_A = 0.
\end{align*}$$  

(4.23a)

(4.23b)

Now, rewrite (4.11b) and (4.16) in the form

$$-\frac{1}{2} dq_A \wedge dq_A = \frac{1}{2} dp_A \wedge dp_A + s^4 \wedge dq_A,$$  

(4.24a)

$$-\frac{1}{2} dq_A \wedge dq_A = \frac{1}{2} dp_A \wedge dp_A + s^4 \wedge dq_A.$$  

(4.24b)

By taking the external “squares” of the both sides of

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Now, externally multiplying (4.30a) and (4.30b) by 
\[ dp^A \wedge dq_A + dp^B \wedge dq_B, \]
respectively, and by applying \( S, \) (4.22), we infer that
\[ ds^A \wedge dq_A + dp^B \wedge dq_B + dr^A \wedge dq_A + dq^B \wedge dq_B = 0, \]
(4.31a)
\[ ds^A \wedge dq_A + dp^B \wedge dq_B + dr^A \wedge dq_A + dq^B \wedge dq_B = 0. \]
(4.31b)

These relations can now be used in (4.25a) and (4.25b) transforming these equations to the form
\[ \gamma = -\left(\frac{1}{2} ds^A \wedge dq_A + dr^A \wedge dq_A + dq^B \wedge dq_B = 0, \right. \]
(4.32a)
\[ \overline{\gamma} = -\left(\frac{1}{2} ds^A \wedge dq_A + dr^A \wedge dq_A + dq^B \wedge dq_B = 0. \right. \]
(4.32b)

Consequently, by applying \( L, \) we infer the existence of functions such that
\[ \frac{1}{2} \Phi^A ds^A + \frac{1}{2} \Phi^B dq_B + A dq_A = d\Lambda. \]
(4.33a)
\[ \frac{1}{2} \Phi_A' ds^A + \frac{1}{2} \Phi_B' dq_B + B dq_A = d\Lambda. \]
(4.33b)

By closing these relations we have of course,
\[ \frac{1}{2} ds^A \wedge dq_A + dr^A \wedge dq_A + dt^A \wedge dq_A = 0, \]
(4.34a)
\[ \frac{1}{2} ds^A \wedge dq_A + dr^A \wedge dq_A + dt^A \wedge dq_A = 0. \]
(4.34b)

Now, if we understand \( \Lambda \) as \( \Lambda = \Lambda(p_A, q_A). \) then from (4.33a) we have
\[ \frac{\partial \Lambda}{\partial p_A} = A = \frac{1}{2} \frac{\partial \Phi^A}{\partial p_A} \frac{\partial \Phi^B}{\partial q_B} \]
so that (4.26a) implies
\[ \frac{\partial \Phi^A}{\partial p_A} + \frac{1}{2} \frac{\partial \Phi^B}{\partial q_B} \frac{\partial \Phi^B}{\partial q_B} = \frac{\partial \Lambda}{\partial p_A}, \]
(4.35)

Now due to the identity \( \partial/\partial p_A)\partial/\partial q_B\) = 0, one easily sees that Eqs. (4.35) imply and are implied by the second heavenly equation (3.5b). Notice that (4.35) is just the spinorial version of the last two equations of Eqs. (2.33) of Ref. 5, which appear in the master equation for determining the Killing vectors in heaven.

Now our procedure of prolongations to obtain new Pfaffian 1-forms can be continued presumably indefinitely. However, there is one important difference. From the 1-forms we have constructed up to now, that is Eqs. (4.17), (4.26), and (4.33), we can choose any neighboring pair (4.33a) and (4.26a), (4.26a) and (4.10b), or (4.10b) and (4.10a), or the corresponding barred pairs to reconstruct the entire heavenly structure. This, however, appears not to be the case as we continue further up the ladder. That is, if we construct the next 1-form and its barred associate by applying the same techniques as previously we cannot use this 1-form in conjunction with (4.33a) to derive the second heavenly equation or its associated 1-forms (4.26a) or (4.10b). It appears that above the \( \Lambda \) 1-form (4.33a) infinitely many 1-forms appear and that possibly all are needed to regain the entire structure. We will now write our heavenly hierarchy of 1-forms in a much more concise notation and also assign complex dilatation weights to the variables which appear.

We begin by noticing that all the 1-forms constructed so far enjoy a scale invariance of the following type:
\[ q_A = \lambda \exp(-4\mu/2)q_A, \quad p_A = -\lambda \exp(i\mu/2)p_A, \]
\[ s_A = \lambda \exp(3i\mu/2)s_A, \quad r_A = -\lambda \exp(5i\mu/2)r_A, \quad (4.36) \]
\[ f_A = \lambda \exp(7i\mu/2)f_A, \quad \Omega = \lambda^2\Omega, \quad \Sigma = \lambda^2\exp(i\mu/2)\Sigma, \]
\[ \theta = \lambda^2\exp(2i\mu)\theta, \quad \Lambda = \lambda^2\exp(3i\mu)\Lambda, \]

where \( \lambda, \mu \in \mathbb{C} \). The corresponding transformations for the barred quantities can be obtained from (4.36) by putting a bar on the corresponding variables and changing \( \mu \rightarrow -\mu \).

This invariance exhibits the fact that all our 1-forms can be characterized by their weights with respect to \( \exp(i\mu) \). This suggests the following change of notation. We introduce spinors \( \phi_A(j) \) and scalars \( \Phi(l) \) defined by

\[ \phi_A(-\frac{1}{2}) = q_A, \quad \overline{\phi}_A(\frac{1}{2}) = q_A, \quad \Phi(0) = -\Omega, \]
\[ \phi_A(\frac{1}{2}) = p_A, \quad \overline{\phi}_A(-\frac{1}{2}) = p_A, \quad \Phi(1) = \Sigma, \quad \Phi(-1) = \overline{\Sigma}, \]
\[ \phi_A(\frac{3}{2}) = s_A, \quad \overline{\phi}_A(-\frac{3}{2}) = s_A, \quad \Phi(2) = \Theta, \quad \Phi(-2) = \overline{\Theta}, \quad (4.37) \]
\[ \phi_A(\frac{5}{2}) = r_A, \quad \overline{\phi}_A(-\frac{5}{2}) = r_A, \quad \Phi(3) = \Lambda, \quad \Phi(-3) = \overline{\Lambda}, \]
\[ \phi_A(\frac{7}{2}) = f_A, \quad \overline{\phi}_A(-\frac{7}{2}) = f_A. \quad (4.38b) \]

It is understood here that \( j \) is a half-odd integer, while \( l \) is an integer. Then we can extend \( \phi_A(j) \) to all negative half-odd integers and \( \overline{\phi}_A(j) \) to all positive half-odd integers by

\[ \phi_A(j) = 0 \quad j < -\frac{1}{2}, \quad (4.38a) \]
\[ \overline{\phi}_A(j) = 0 \quad j > \frac{1}{2}. \quad (4.38b) \]

Now the important point is that we can apparently also extend \( \phi_A(j) \), \( \overline{\phi}_A(j) \) to the remaining half-odd integers and \( \Phi(l) \) to all integers by the prolongation process. Indeed using (4.37) and (4.38), we can write all our previous Pfaffian 1-forms (4.17), (4.26), and (4.33) succinctly as

\[ \phi(l) = \frac{1}{2} \sum_j \{ \phi^A(l-j) \, d\phi_A(j) + \overline{\phi}^A(l-j) \, d\overline{\phi}_A(j) \}, \quad (4.39) \]

where \( j \) runs over all half-odd integers. Now the previously obtained 1-forms are given by the range \( l = 3, \ldots, 3 \). However, we have checked the validity of (4.39) for the larger range \( l = -7, \ldots, 7 \). Indeed it appears that (4.39) is valid for all integers \( l \). Hence, we conjecture that the heavenly hierarchy given by (4.39) is, in fact, infinite. We have not been able to prove our conjecture, however. One might think that an inductive proof would work, but a closer examination shows that one must invoke the induction hypothesis at each stage of the prolongation process, i.e., it is necessary to alternate invoking the induction hypothesis with implementing Lemma L. In spite of this we see no reason why the prolongation process should break down for higher values of \( l \).

Now the closure relations for (4.39) are given by

\[ \omega(l) = \frac{1}{2} \sum_j \{ d\phi^A(l-j) \wedge d\phi_A(j) + d\overline{\phi}^A(l-j) \wedge d\overline{\phi}_A(j) \} = 0, \quad (4.40) \]

It is clear that the relations (4.40) split in a natural fashion into three subfamilies: the purely heavenly subfamily (no dotted spinors)

\[ \omega(l) = \frac{1}{2} \sum_j d\phi^A(l-j) \wedge d\phi_A(j) = 0, \quad l \geq 2; \quad (4.41) \]

the purely hellish subfamily (no undotted spinors)

\[ \omega(l) = \frac{1}{2} \sum_j d\phi^A(l-j) \wedge d\overline{\phi}_A(j) = 0, \quad l \geq 2; \quad (4.42) \]

and the subfamily where the three forms, \( d\phi(1), d\phi(0), \) and \( d\phi(-1) \) necessarily mix the undotted spinors. The corresponding equations are of course

\[ \omega(l) = \frac{1}{2} d\phi^A(\frac{l}{2}) \wedge d\phi_A(-\frac{l}{2}) + d\overline{\phi}^A(\frac{l}{2}) \wedge d\overline{\phi}_A(-\frac{l}{2}) \]

\[ + \frac{1}{2} d\phi^A(-\frac{l}{2}) \wedge d\phi_A(\frac{l}{2}) = 0, \quad (4.43) \]

\[ \omega(0) = d\phi^A(\frac{1}{2}) \wedge d\phi_A(-\frac{1}{2}) + d\overline{\phi}^A(\frac{1}{2}) \wedge d\overline{\phi}_A(-\frac{1}{2}), \]

\[ + \frac{1}{2} d\phi^A(-\frac{1}{2}) \wedge d\phi_A(\frac{1}{2}) = 0. \]

It is quite clear that \( \omega^* \) or \( \omega^* \) assumed is enough to reproduce all the structure considered.

Finally we mention that from (3.12) one easily sees that equalities (4.43) amount to the description of the forms \( s^{AB} \) through the alternative formulas

\[ s^l = d\phi^A(-\frac{1}{2}) \wedge d\phi_A(\frac{l}{2}) \]

\[ = -2 d\phi^A(-\frac{1}{2}) \wedge d\overline{\phi}^A(-\frac{1}{2}) - d\overline{\phi}_A(-\frac{1}{2}) \wedge d\overline{\phi}_A(-\frac{1}{2}), \quad (4.44) \]

\[ s^l = -d\phi^A(\frac{l}{2}) \wedge d\phi_A(-\frac{l}{2}) = d\phi^A(-\frac{l}{2}) \wedge d\phi_A(\frac{l}{2}) \]

\[ = 2 d\phi^A(\frac{1}{2}) \wedge d\phi_A(-\frac{1}{2}) \]

\[ + d\phi^A(-\frac{1}{2}) \wedge d\phi_A(\frac{1}{2}). \]

5. General Properties of the Integral Manifolds

In this section we consider some important properties of the heavenly integral manifolds. While we have not been able to find an explicit expression for the general solution, we can use Cartan's theory, to construct stepwise the regular integral manifolds in terms of their tangent spaces. This will allow us, for example, to determine at each step the arbitrariness of the integral manifolds, i.e., on how many arbitrary functions of how many variables the general manifold depends. To do this we can begin with any of the equivalent forms of the heavenly manifolds. It seems best to use the already partially integrated description given in terms of the two Pfaffian 1-forms (4.10b) and (4.26a). Here we write them in component form

\[ \omega_1 = d\Theta + s_2 d\varphi_1 - s_1 d\varphi_2 + \varphi_1 d\varphi_2 - \varphi_2 d\varphi_1, \quad (5.1) \]

\[ \omega_2 = d\Xi + \frac{1}{2} s_2 d\varphi_1 - \frac{1}{2} s_1 d\varphi_2 + \frac{1}{2} \varphi_1 d\varphi_2 - \frac{1}{2} \varphi_2 d\varphi_1 + \frac{1}{2} p_2 d\varphi_1 - \frac{1}{2} p_1 d\varphi_2. \]

These forms are, of course, zero on an integral manifold. The closure easily gives

\[ d\omega_1 = d\omega_1 \wedge d\varphi_1 - d\omega_1 \wedge d\varphi_2 + d\varphi_1 \wedge d\varphi_2 \wedge d\varphi_1 \wedge d\varphi_2, \quad (5.2) \]

\[ d\omega_2 = d\omega_2 \wedge d\varphi_1 - d\omega_2 \wedge d\varphi_2 + d\varphi_1 \wedge d\varphi_2 \wedge d\varphi_1 \wedge d\varphi_2. \]

Now we are working on complex Euclidean \( n \)-space where \( n = 12 \). We wish to find the regular integral manifolds by successive applications of the Cauchy-Kowalewski theorem. Now at a regular point in \( \mathbb{C}^{12} \), the rank of the system (5.1) is 2, so the Cartan character \( s_2 = 2 \). A vector in the tangent plane to a solution must satisfy

\[ X \cdot \omega_1 = X \cdot \omega_2 = 0 \]

where \( X \) denotes the inner product between differential.
forms and vector fields. The polar system is obtained by adjoining to (5.1) the 1-forms

\[ X_1 \downarrow d\omega_1, \quad X_j \downarrow d\omega_j, \quad (5.4) \]

where \( X \) satisfies (5.3). This system has rank 4 (i.e., \( s_0 + s_1 = 4 \)), so \( s_1 = 2 \). A two-dimensional integral manifold is obtained by constructing \( X_2 \) to satisfy (5.3) and

\[ X \downarrow (X_1 \downarrow d\omega_1) = X \downarrow (X_j \downarrow d\omega_j) = 0, \quad (5.5) \]

Its polar system is obtained by adding to (5.1) and (5.4) the 1-forms \( X_2 \downarrow d\omega_1 \) and \( X_2 \downarrow d\omega_2 \) which has rank 6 and thus \( s_2 = 2 \). Continuing in this way we obtain a three-dimensional integral manifold with tangent vectors \( \{X_1, X_2, X_3\} \), where \( X_3 \) satisfies (5.3), (5.5), and (5.5) with \( X_1 \) replaced by \( X_2 \). The polar system is obtained by adding \( X_2 \downarrow d\omega_1 \) and \( X_2 \downarrow d\omega_2 \) to the previous polar system. Its rank is 8, thus \( s_2 = 2 \). The four-dimensional integral manifolds \( \{X_1, X_2, X_3, X_4\} \) are constructed as before with \( X_4 \) orthogonal to the last constructed polar system. However, if we add \( X_4 \downarrow d\omega_1 \) and \( X_4 \downarrow d\omega_2 \) to this polar system, its rank remains 8, since we are in \( \mathbb{C}^2 \) and \( 12 - 4 = 8 \). Thus the genus \( g = 4 \) and the maximal regular integral manifolds have complex dimension four, which of course we already knew. We also have \( s_4 = 0 \).

Now we can use Cartan’s criteria (Ref. 8, p. 75) to state for example, that the general solution for heavens depends on two arbitrary functions of three complex variables. To sum up, the regular maximal integral manifolds of heaven are determined by

\[ X_1 \downarrow \omega_1 = X_i \downarrow \omega_i = 0, \quad (5.6) \]

\[ X_1 \downarrow (X_j \downarrow d\omega_1) = X_j \downarrow (X_i \downarrow d\omega_i) = 0, \quad i \neq j, \]

\( i,j = 1, \ldots , 4 \). The only qualification that we must add is that (4.4) be satisfied, i.e., that the system (5.1) be in involution with respect to the spinors \( q_A, p_A \). Equations (5.6) give, at least implicitly, the general integral manifolds of heaven in terms of the tangent spaces at each point.

In order to find explicit integral manifolds for heaven, we deal with the second heavenly equation in the form given by (4.25a) and the 2-form \( d\omega_t \), given by (5.2). In component form, (4.25a) reads

\[ ds_1 \wedge ds_2 \wedge dq_1 \wedge dq_2 + ds_2 \wedge dq_1 \wedge dp_1 \wedge dp_2 - ds_1 \wedge dq_2 \wedge dp_1 \wedge dp_2 = 0, \quad (5.7) \]

on an integral manifold. In fact we would like to be able to linearize, at least partially, the differential equations for an integral manifold. Indeed we will see that the case treated in detail in Sec. 2 is a special case of the linearization that follows. Again as in Sec. 2, the trick is to integrate the equation \( d\omega_t = 0 \), treating \( q_1, q_2, p_1, \) and \( s_1 \) as independent variables. We then exist a function \( \Psi(q_1, q_2, q_3) \) with

\[ d\Psi = -s_2 dp_1 - p_2 ds_1 + r_2 dq_1 + r_1 dq_2 = 0 \quad (5.8) \]

on an integral manifold. Plugging (5.8) back into (5.7) we obtain the differential equation

\[ \Psi_{s_2} q_2 + \Psi_{r_2} q_1 + \Psi_{r_1} q_2 + \Psi_{q_1} q_1 - \Psi_{p_1} q_1 = 0. \quad (5.9) \]

The advantage of this form is the following: As seen from (2.7), the derivatives of \( \Psi \) with respect to \( q_1 \) do not enter into the calculation of the metric, and thus it is also this way with \( \Psi \). Moreover, the nonlinear terms in (5.9) contain derivatives with respect to the \( q_A \)'s. Thus (5.9) is susceptible to linearization involving nontrivial metrics. We mention that the condition that the original variables \( p_1, q_2 \) are independent (i.e., \( dV \neq 0 \)) imply that \( \Psi_{s_2} q_2 \neq 0 \), and that only those heavenly manifolds such that \( \Psi_{s_2} q_2 = 0 \) are amenable to the above treatment. The case \( \Psi_{s_2} q_2 \neq 0 \) is easily handled, however, as shown in Ref. 3.

Now it is straightforward to determine the metric in terms of the function \( \Psi \). Indeed, the necessary derivatives are

\[ \Theta_{s_2} q_2 = \Psi_{s_2} q_2, \quad \Theta_{r_2} q_1 = -\Psi_{r_1} q_1, \quad \Theta_{r_1} q_2 = \Psi_{r_1} q_2. \quad (5.10) \]

Simple substitution of (5.10) into (3.7) then gives the spinorial components of the metric in the \( \Psi \) formalism. Similarly the connection and curvature components can be computed; however, we do not give these explicitly. The important point is that as in the \( \phi \) formalism, the metric, connections, and curvature do not involve derivatives with respect to \( q_A \).

With this in mind we look for solutions of (5.9) with \( \Psi_{s_2} q_2 = 0 \). [The counterpart in the \( \phi \) formalism is \( \Theta_{s_2} q_2 = \Phi_{s_2} q_2 = 0 \) which does not enter the second heavenly equation (3.5b) explicitly.] Thus \( \Psi \) can be written as \( \Psi = F(q_1, q_2, q_3) + \delta(q_1, q_2, q_3) \) and the analysis of (5.9) splits into two cases depending on whether \( F(q_1, q_2) \) vanishes or not.

Case 1: \( F(q_1, q_2) \neq 0 \).

This case reduces to the three dimensional complex Laplace equation after some gauging and changes of variables,

\[ \Psi_{s_2} q_2 + \Psi_{r_2} q_1 + \Psi_{r_1} q_2 = 0, \quad (5.11) \]

where now \( \Psi \) is a function of \( p_1, q_2, p_4 \), and

\[ p_2 = q_1 + \zeta, \quad p_4 = q_4 + i\zeta, \]

\[ \zeta = s_1 + \int \alpha(q_1) dq_1, \]

\[ \alpha(q_1) := \psi_{s_2} q_2. \]

There are many ways, of course, to solve (5.11) depending on different domains of holomorphy. The general solution can be given explicitly and depends on two holomorphic functions of two complex variables. We mention in addition, regarding solution techniques for (5.11), Ref. 13 where group theoretical techniques are used and Ref. 14 where an operational calculus approach is used. We also mention that the special case when \( \alpha(q_1) = 1 \) and \( \Psi \) is independent of \( q_1 \) reduces to the two-dimensional Laplace equation treated in Sec. 2.

Case 2: \( F(q_1, q_2) = 0 \).

This case has two vanishing conformal curvature components, i.e., \( C^{(5)} = C^{(6)} = 0 \). The quantities necessary to compute the metric are

\[ \Psi_{s_2} q_2 = \delta(q_1) p_1 + \delta(q_1), \]

\[ \Psi_{r_2} q_1 = \delta(q_1) s_1 + \delta(q_1) p_1 + \delta(q_1), \quad \Psi_{r_1} q_2 = \delta(q_1) s_1 + \delta(q_1) p_1 + \delta(q_1), \quad (5.12) \]

\[ \Theta_{s_2} q_2 = \Psi_{s_2} q_2 + \frac{1}{2} \delta(q_1) p_1 + \frac{1}{2} \delta(q_1) p_1 + f'_{s_2} q_2 + f'_{r_2} q_1 + \Psi_{r_1} q_2, \]

\[ f'_{s_2} q_2 + f'_{r_2} q_1 + \Psi_{r_1} q_2. \]
where \( g^A, f^A, r^A \) are arbitrary functions of \( q_1 \), \( h \) is an arbitrary function of its arguments, and \( \xi := q_A - \frac{1}{2} g^B (p_B + g^A / g^B)^2 \). This case has some overlap with the case treated beginning with (4.12a) in Ref. 5, but in general they are not equivalent.

It is clear from the above analysis that many classes of metrics appear and can be given explicitly and hence studied in much more detail along the lines of Sec. 2. We will not do this here, however. Finally, it is mentioned that a similar linearization yielding non-trivial metrics is obtained by setting \( \Psi_{A T} = 0 \) in (5.9).

6. SYMMETRIES OF THE SECOND HEAVENLY EQUATION

In this section we describe the symmetries of the second heavenly equation. In fact, we show that essentially the infinitesimal symmetries coincide with the Killing vectors obtained in Ref. 5 aside from the function \( \Lambda \), which we have already seen arises from the prolongation process described in Sec. 4. Generally it would be of interest to study the symmetries of the complete heavenly hierarchy or at least the system of 1-forms which begin with and end with \( \Lambda \), i.e., \( \ldots, \Lambda, \Lambda^0, \Lambda^1, \ldots, \Lambda^2, \ldots \). This could shed light on the meaning of the hierarchy. However, we content ourselves here with finding the infinitesimal symmetries of the pair of Pfaffian 1-forms (4.26a) and (4.33a). The reason for choosing here the 1-form (4.33a) instead of (4.10b) to represent the heavenly integral manifolds is that it allows for the dependence of the symmetries on \( \Lambda \) which is of interest from the point of view of the Killing vectors. We will show, however, that this dependence is not allowed as transformations on the space spanned in a local chart by \( (q_A, p_A, \phi) \).

Now let \( M \) be a differential manifold and let \( J \) be an ideal in the Grassmann algebra \( \Lambda(M) \). Also let \( J \) be closed under exterior differentiation. Suppose that \( J \) is generated by \( \omega \) and \( d\omega \) and let \( (N, i) \) be an immersed submanifold which annuls \( J \), i.e., an integral manifold. Then the local symmetry group for \( J \) is given by the set of all local diffeomorphisms \( \phi : M \rightarrow M \) such that

\[
\phi^* \omega \in J \quad \text{for all} \quad \omega \in J,
\]

(6.1)

where \( \phi^* \) denotes the pullback of \( \phi \). Hence \( \phi \) is a mapping on \( M \) such that the integral manifolds of \( J \) are mapped into themselves. Infinitesimally (6.1) reads

\[
\varepsilon_x \omega = \lambda^i \omega_i,
\]

(6.2)

for all \( \omega \in J \) and where \( \omega_i \in \Lambda^i \), \( \lambda^i \) are locally holomorphic functions on \( M \), and \( X \) is the vector field describing a local one-parameter trajectory \( \phi_t \).

Now let us apply (6.2) to the ideal \( J \) of differential forms generated by the two 1-forms (4.26a), (4.33a), and their closures (4.30a), (4.34a). Explicitly, we write

\[
\omega_1 = d\phi - s^A dp_A - r^A dq_A,
\]

(6.3a)

\[
\omega_2 = d\Lambda - \frac{1}{2} s^A dq_A - r^A dp_A - t^A dq_A,
\]

(6.3b)

Then applying (6.2) we have

\[
\varepsilon_x \omega = \lambda^i \omega_i + \lambda^d \omega_d, \quad \text{(6.4a)}
\]

We mentioned that the commutivity of the Lie derivative and the exterior derivative applied to (6.4) guarantees that the 2-forms \( d\omega_1 \) and \( d\omega_2 \) are back in \( J \) after an infinitesimal transformation. Now in order to solve (6.4) we write the \( \omega_i \) out explicitly and make use of the identity

\[
\varepsilon_x \omega = d(X \cdot \omega) + X \cdot d\omega.
\]

Then equating the coefficients of the independent 1-forms on the space \( G^{12} \) we obtain a system of first order coupled partial differential equations for the vector fields \( X \). In order to facilitate matters it is convenient to define functions \( F \) and \( G \) by

\[
F = X \cdot \omega_1, \quad G = X \cdot \omega_2.
\]

(6.5)

Then upon equating coefficients in (6.4) we obtain the equations

\[
F_1 = 0, \quad X^A = F_{\tau A} = G_{\tau A},
\]

(6.6)

\[
X^A = F_{\tau A} + s^A F_{\theta} + r^A F = G_{\tau A} + \frac{1}{2} s^A G_A,
\]

(6.7)

\[
X^A = F_{\tau A} + r^A F_{\theta} + s^A F = G_{\tau A} + s^A G_A + r^A G_A,
\]

(6.8)

where \( X^A \) denotes the component of \( X \) multiplying \( \omega^A \), etc. The first three of Eqs. (6.7) can be integrated immediately to give

\[
\begin{align*}
F &= F^A_{\tau A} + F^\theta, \\
G &= G^A_{\tau A} + \frac{1}{2} s^A F_A + G_{\theta A},
\end{align*}
\]

(6.9)

where \( F^A, F^\theta, \) and \( G_{\theta A} \) are arbitrary functions of the spinors \( q_A, p_A, s_A \) and scalars \( \tau, \theta, \) \( \Lambda \). Plugging (6.8) into the last two of Eqs. (6.7) and doing some algebra, we have

\[
\begin{align*}
F_0 &= 0, \quad F_0 = H^A s_A + H^\theta, \\
F_0 &= F^A_{\theta A} = F^\theta = F_{\theta A} = 0, \\
F_A &= s^A F_A + G^\theta + \frac{1}{2} s^A G_A, \\
G^\theta &= G^\theta + \frac{1}{2} s^A G_A + H^\theta, \\
F^A_{\theta A} &= e^A \xi - H_{\theta A} + H_{\theta A},
\end{align*}
\]

(6.10)

where \( H^A \) and \( H^\theta \) are arbitrary functions of \( q_A, p_A, \) \( \tau, \) and \( \Lambda \). The integration of (6.9) is straightforward but rather tedious. First we notice that \( F^\theta = 0 \) is a function only of \( q_A \) and after some algebra we find that \( H^A, H^\theta, \) and \( G_{\theta A} \) must have the forms

\[
\begin{align*}
H^A &= H^A_{BC}(q) \partial_C + H^A(q), \\
H^\theta &= H^\theta(q) \partial + h^\theta(q, p), \\
G^\theta &= G^\theta(q) \partial + g^\theta(q, p, s),
\end{align*}
\]

(6.11)

and we are left with the constraint equations

\[
\begin{align*}
\xi^A &= \frac{1}{2} F^A_{\theta A} s_A + h^\theta s_A, \\
H^A &= F^A_{\theta A} + e^A \xi, \\
G^A_{\theta A} &= (H^A + D^A_{\theta A} - e^A \xi s_A) c \xi + (H^\theta_{\theta A} - \alpha^A e^A s_A + h^\theta_{\theta A}),
\end{align*}
\]

(6.12)

where \( h^\theta \) and \( \alpha^A \) are functions of \( q_A \) only. Upon further integrations of (6.11) we find that both \( h^\theta \) and \( \alpha^A \) must be constants and

\[
F^A = \phi^A + C_A e^A q_A.
\]

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\[ H^{AB} = \phi_{A}^{A'B} + C_{B}^{A} q_{A}, \]
\[ H^{\alpha 0} = \psi_{\alpha} - a_{\alpha} q_{B}, \]
\[ \dot{h}^{\alpha} = \frac{1}{2} \phi_{A}^{A'B} q_{B} + \frac{1}{2} \psi_{\alpha} q_{\alpha} + h^{\alpha 0}(q), \]
\[ \dot{g}^{0} = \frac{1}{2} \phi_{A}^{A'B} q_{B} + \frac{1}{2} \psi_{\alpha} q_{\alpha} + h^{0}(q). \]

(6.12)

This ends the computation of the infinite dimensional Lie algebra \( \mathcal{L} \) of infinitesimal symmetries of the heavenly manifolds. It is not difficult to see that the only symmetry in (6.12) which is not a projection onto transformations of the space with local coordinates \((q_{A}, p_{A}, \theta)\) is that symmetry generated by the function \( g^{0}(q) \). This function generates \( q_{A} \) dependent translations of \( \Lambda \) and leaves (6.3b) invariant since the spinor \( t_{A} \) is essentially arbitrary. The Lie algebra \( \mathcal{L}_0 \) generated by these translations is therefore less interesting. Indeed, it can be seen that \( \mathcal{L}_0 \) is an ideal in \( \mathcal{L} \), and we thus consider the factor algebra \( \mathcal{L}/\mathcal{L}_0 \). The projections of these onto the base space spanned by \((q_{A}, p_{A}, \theta)\) are given by the vector fields

\[ X^{\alpha} = \phi_{A}^{A} + C_{B}^{A} q_{B}, \]
\[ X^{A} = \phi_{A}^{A} p_{B} + C_{B}^{A} p_{A} + h^{A}(q), \]
\[ X^{0} = (C_{B} + 3C_{0}) \theta + h^{0}. \]

(6.13)

It is now easy to see that the vector fields (6.13) are precisely the Killing vectors in spinorial notation given by Eq. (2.33) of Ref. 5, with \( \alpha_0 = 0 \). On the other hand, we have understood the \( \alpha_0 \) term in terms of the prolongation variable \( \Lambda \) in Sec. 4.

Finally, we mention the possible use of the symmetries to obtain solutions of the second heavenly equation. Given a symmetry vector field we can find relative invariants which essentially reduces the number of variables of the original partial differential equation by one. Indeed if we know three independent symmetries we can reduce the problem to quadratures. Moreover, given any solution we can obtain other solutions by group transformations.

Closing this paper, we should like to conclude that we believe that its results, although not as complete as one might desire, seem to justify our belief that (i) it is profitable to use an abbreviated spinorial notation as introduced in Sec. 3, and (ii) that the apparatus of the canonical Cartan's theory of integral manifolds is suitable when striving towards better understanding of the nature of heavens.

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