Electromagnetic and gravitational Hertz potentials

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Electromagnetic and gravitational Hertz potentials

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With generality of complex relativity, the classical theory of the electromagnetic Hertz potentials is outlined in terms of spinors and forms. Particularly interesting are $D(0,1)$ and $D(1,0)$ null Hertz potentials. Then, a new spinorial approach to heavens (ℍ spaces) is proposed, which reveals in the structure the presence of the left null gravitational Hertz potential [of the type $D(0,2)$]. The relevant hints which follow from our results and concern the structure of the most general solutions of the Einstein vacuum equations (type $G ⊗ G$), are discussed, in particular on the level of the linearized theory.

1. PRELIMINARIES AND THE FORMALISM USED

A fairly complete review of various approaches to electromagnetic Hertz potentials (which also includes the treatment employing differentials forms) can be found in the excellent article by Cohen and Kegels; the older pertinent references can be also localized there.

Our aim is to propose a new approach to the dynamical equations of nonlinear general relativity, founded on an appropriately generalized notion of Hertz potentials. For this purpose, we will first outline in this section the spinorial description of the Riemannian geometry of a complex space-time. Then, the classical theory of Hertz potentials and the basic results of the theory of heavens established in Ref. 2 will be examined in the light of the spinorial formalism. This will lead in the subsequent sections to some general ideas about the gravitational Hertz potentials.

Thus, we will work in a complex (Riemannian) space-time which is a pair: a (complex) analytic manifold $M$ and the metric

$$g = -\frac{1}{2} \varepsilon_{AB} \varepsilon^{CD} \xi^A \xi^B$$

(1.1)

where, labelled by the spinorial indices ($A = 1, 2; B = 1, 2$), $g_{AB} \in \Lambda^1$ form the base of the cotangent space. The spinorial indices are manipulated by Levi-Civita’s symbols according to the usual conventions, e.g.,

$$\varepsilon_{AB} \varepsilon^{CD} = \varepsilon_{CD} \varepsilon^{AB}.$$

The gauge group of the theory $G = \text{SL}(2,\mathbb{C}) \times \text{SL}(\mathbb{C},\mathbb{C})$ in an obvious symbolism. In real relativity, $\text{SL}$ transformations are complex conjugates of $\text{SL}$, $\text{SL} = (\text{SL})^*$ and $\text{SL}$ maintains the hermicity of $g_{AB}$—and thus the signature $(++-)$ of the real metric over the real manifold $M$. In complex relativity, the two copies of the $\text{SL}(2,\mathbb{C})$ group, $\text{SL}$ and $\text{SL}^*$, remain independent.

In the space of the multiforms, $\Lambda = \oplus_{\alpha} \Lambda^\alpha$, we have two basic mappings: the external differential and the Hodge star

$$d: \Lambda^\alpha \rightarrow \Lambda^{\alpha + 1}, \quad d^2 = 0,$$

$$*: \Lambda^\alpha \rightarrow \Lambda^{\alpha - 1}, \quad ** = d \ast.$$

(1.2a)

(1.2b)

We will also employ the concept of the codifferential defined by

$$\delta = -i d^*; \quad \Lambda^\alpha \rightarrow \Lambda^{\alpha - 1}, \quad \delta^2 = 0.$$

(1.3)

For our present purposes, it is also convenient to work with the concept of the Laplace-Beltrami operator

$$\Delta = d \delta + \delta d; \quad \Lambda^\alpha \rightarrow \Lambda^\alpha,$$

(1.4)

called subsequently the “Laplacian,” and the associated operator

$$\Delta = d \delta + \delta d; \quad \Lambda^\alpha \rightarrow \Lambda^\alpha,$$

(1.5)

called subsequently the “anti-Laplacian.” We have, of course,

$$d \delta = \frac{1}{2} (\Delta - \Delta), \quad \delta d = \frac{1}{2} (\Delta + \Delta),$$

(1.6)

and then

$$\Delta \Delta - \Delta \Delta = 0,$$

(1.7a)

$$\Delta^2 = (d \delta)^2 + (\delta d)^2 = \Delta.$$

(1.7b)

The name “anti-Laplacian” is perhaps justified by the fact that:

$$\Delta \Delta - \Delta \Delta = 0,$$

(1.8a)

$$\Delta \delta - \delta \Delta = 0,$$

(1.8b)

$$\Delta = \Delta.$$

The formalism which we use employs, with respect to $\Lambda^\alpha$-valued spinors, the covariant differential $D: \Lambda^\alpha \rightarrow \Lambda^{\alpha + 1}$ defined according to

$$D \xi^\alpha = d \xi^\alpha + \Gamma^\alpha_\beta \wedge \xi^\beta + \Gamma^\alpha_\gamma \wedge \xi^\gamma + \cdots,$$

(1.9)

where $\Gamma^\alpha_\beta = \Gamma^\alpha_\beta$ and $\Gamma^\alpha_\beta = \Gamma^\alpha_\beta$ are respectively the left and right connection 1-forms. The connection forms we understand as determined by the $g_{AB}$ via the first structure equations:

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\[
D_A^{AB} = d g^{AB} = \Gamma^A_{\beta\gamma} g^{\beta B} + \Gamma^B_{\beta\gamma} g^{A\gamma} - \Gamma^A_{\beta\gamma} g^{B\gamma} = 0. \quad (1.10)
\]

Now the covariant derivatives of the connections, which are \( g \)-tensors, determine at the same time the left and right curvature forms:
\[
\begin{align*}
DT^{AB} &= dT^{AB} + \Gamma^A_{\beta\gamma} T^{\beta B} + \Gamma^B_{\beta\gamma} T^{A\gamma} = R^A_{\beta\gamma}, \\
DR^{AB} &= dR^{AB} + \Gamma^A_{\beta\gamma} R^{\beta B} + \Gamma^B_{\beta\gamma} R^{A\gamma} = R^A_{\beta\gamma} \quad (1.11a, b)
\end{align*}
\]

The Bianchi identities, which are integrability conditions for (1.11), are then
\[
DR^A_{\beta\gamma} = 0 = DR^A_{\beta\gamma}, \quad (1.12)
\]

Observe that the general formulas
\[
DDT^{AB...}_{CD...} = R_A^{CD...} \wedge T^{AB...}_{CD...} + R_B^{CD...} \wedge T^{AB...}_{CD...} - R_C^{CD...} \wedge T^{AB...}_{CD...} - R_D^{CD...} \wedge T^{AB...}_{CD...} + \cdots, \quad (1.13)
\]
take in the present formalism the role of the Ricci tensor, and permit us to investigate conveniently the integrability conditions of any equations formulated by the use of the \( D \) operation.

The relation:
\[
g^{AB} \wedge g^{CD} = \delta^A_C \delta^B_D + \delta^A_D \delta^B_C \quad (1.14)
\]
defines the objects,
\[
S_{AB} = * S_{AB} = S_{A[B]} \in \Lambda^2, \quad (1.15a)
\]
\[
S^*_{AB} = - * S^*_{AB} = S^*_{A[B]} \in \Lambda^2, \quad (1.15b)
\]
which form a complete base of \( \Lambda^2 \). Knowing this, and using as a consequence of (1.10) \( DDg^{AB} = 0 \) (i.e., \( R^{\Lambda}_A \wedge g^{\Lambda B} + R^{\Lambda}_B \wedge g^{A\Lambda} = 0 \)), one easily shows that the curvature forms can be always represented as:
\[
R_{AB} = - \frac{1}{2} C_{ABCD} SD + (R/24) S_{AB} + \frac{1}{2} C_{ABCD} SD, \quad (1.16a)
\]
\[
R_{AB} = - \frac{1}{2} C_{ABCD} SD + (R/24) S_{AB} + \frac{1}{2} C_{ABCD} SD, \quad (1.16b)
\]
where the \( D(0,2) \) and \( D(0,2) \) objects, \( C_{ABCD} = C_{(ABCD)} \) and \( C_{[ABCD]} = C_{[ABCD]} \), are the spinorial images of the self-dual and anti-self-dual parts of the conformal curvature tensor \( C_{ABCD} \wedge C_{[ABCD]} \); \( R \) is the scalar curvature; and \( C_{ABCD} = C_{(ABCD)} \) is a \((1,1)\) object which corresponds to the traceless part of the Ricci tensor.

Our formalism employs the concept of the spinorial gradient, \( \partial_{A[B]} \), and of the covariant spinorial gradient \( \nabla_{A[B]} \). Of course, \( \partial_{A[B]} \) can be thought of as the base of the tangent space, and for every \( T_{KLI...} \in \Lambda^A \) we have
\[
dT_{KLI...} = - \frac{1}{2} g^{AB} \partial_{A[B]} T_{KLI...}, \quad (1.17)
\]

The covariant gradient is then defined by a parallel formula:
\[
DT_{KLI...} = - \frac{1}{2} g^{AB} \nabla_{A[B]} T_{KLI...}. \quad (1.18)
\]

Using the operator \( \nabla_{A[B]} \), one easily shows that the Bianchi identities amount to
\[
\begin{align*}
\nabla_{C[AB} S_{CD]} &= \nabla_{A[C} S_{DB]} \delta^B_C = 0, \\
\nabla_{D[C} S_{AB]} &= \nabla_{A[C} S_{DB]} \delta^B_C = 0, \\
\nabla_{A[B} S_{CD]} &= \nabla_{A[C} S_{DB]} \delta^B_C = 0, \quad (1.19a-c)
\end{align*}
\]

The formalism succinctly outlined here, for the sake of completeness, is described more fully in Ref. 3. Further developments and applications of the formalism are discussed in the first section of Ref. 2 and the subsequent papers about heavens, Refs. 4, 5 and particularly 6.

We close this section by establishing our convection for the inner product of forms:
\[
\alpha \wedge \beta = * (\alpha \wedge \beta) \quad (1.20)
\]

which holds with the star so normalized that \( * * = \) identity.

2. THE MAXWELL EQUATIONS AND THE ELECTROMAGNETIC HERTZ POTENTIALS

In a real \( V_4 \), the 2-form of the electromagnetic field \( f = \frac{1}{2} f_{\mu\nu} dx^\mu \wedge dx^\nu \) (in terms of the local components) has to fulfill the Maxwell (vacuum) equations
\[
df = 0 = df \quad (2.1)
\]

and is supposed to be real. It can be always decomposed into the pure self-dual and anti-self-dual parts:
\[
\begin{align*}
\omega &= f + * f = 2 f_{AB} S^{AB}, \\
\overline{\omega} &= f - * f = 2 f_{A'B'} * S^{A'B'},
\end{align*}
\]

where \( f_{AB} \) and \( f_{A'B'} \), the spinorial images of the electromagnetic field, can be thought of as objects of helicity \( + \) and \( - \) respectively. The objects \( \omega \) and \( \overline{\omega} \) then satisfy
\[
\begin{align*}
\star \omega &= \omega, & d\omega &= 0, & \delta \omega &= 0, \\
\star \overline{\omega} &= \overline{\omega}, & d\overline{\omega} &= 0, & \delta \overline{\omega} &= 0,
\end{align*}
\]

In a real \( V_4 \), \( \overline{\omega} \) is complex conjugate of \( \omega \). In a complex \( V_4 \), \( \omega \) and \( \overline{\omega} \) became independent objects and (2.3a) and (2.3b) are respectively the left ("heavenly") and the right ("hellarish") Maxwell equations. Notice that with \( \omega \) and \( \overline{\omega} \) being of definite helicity, it is enough to assume that either the differential or codifferential of \( \omega \) and \( \overline{\omega} \) vanish; the others then vanish automatically. Notice also that, as a consequence of (2.3), we have
\[
\begin{align*}
\partial \omega &= 0, & \overline{\partial} \omega &= 0, \\
\overline{\partial} \overline{\omega} &= 0, & \overline{\partial} \overline{\overline{\omega}} &= 0, \quad (2.4a, b)
\end{align*}
\]

We can now state the basic idea of the Hertz potentials as follows: Let \( \Pi, \overline{\Pi} \in \Lambda^2 \) be forms which have a definite helicity and are solutions of the harmonic equation:
\[
\begin{align*}
\star \Pi &= \Pi, & \partial \Pi &= 0, \\
\star \overline{\Pi} &= \overline{\Pi}, & \overline{\partial} \overline{\Pi} &= 0.
\end{align*}
\]

Then, having such forms, we can construct solutions of the left and right Maxwell equations by writing
\[
\omega = \partial \Pi \overline{\partial} \Pi - \overline{\partial} \Pi \partial \Pi = \overline{\star} \Pi, \quad (2.6a)
\]
\[
\overline{\omega} = \overline{\partial} \Pi \overline{\overline{\partial}} \Pi - \overline{\overline{\partial}} \Pi \overline{\partial} \Pi = \star \Pi, \quad (2.6b)
\]

Indeed, the differentials and coderivatives of both \( \omega \) and \( \overline{\omega} \) then vanish as the consequence of \( \partial^2 = 0 = \overline{\partial}^2 \) and we have \( \star \omega = \omega \) and \( \star \overline{\omega} = \overline{\omega} \), as the consequence of \( \star \star = 0 \) and the assumed helicities for \( \Pi \) and \( \overline{\Pi} \).

One can easily show that every left and right electromagnetic field can be always represented through the corresponding Hertz potentials and, moreover, that
there remains a great deal of ambiguity with which these potentials are defined when $\omega$ and $\bar{\omega}$ are given. One easily shows that the formulas (2.5) and (2.6) stay unchanged with respect to such a gauge of II's that

\[ \Pi = \Pi + \nu, \quad \Delta \nu = 0 = \Delta \nu, \quad (2.7a) \]
\[ \bar{\Pi} = \bar{\Pi} + \bar{\nu}, \quad \Delta \bar{\nu} = 0 = \Delta \bar{\nu}, \quad (2.7b) \]

i.e., the gauge forms $\nu, \bar{\nu} \in \Lambda^2$ being the solutions of the inhomogeneous left and right Maxwell equations,

\[ *\nu = \nu, \quad d\nu = i*\delta \mu, \quad d\nu = \delta \mu, \quad (2.8a) \]
\[ *\bar{\nu} = \bar{\nu}, \quad d\bar{\nu} = i*\bar{\delta} \bar{\mu}, \quad d\bar{\nu} = \bar{\delta} \bar{\mu}, \quad (2.8b) \]

where $\mu, \bar{\mu} \in \Lambda^0$ are arbitrary harmonic functions:

\[ \Delta \mu = 0 = \Delta \bar{\mu}, \quad (2.9) \]

It is of interest to describe all this in terms of $\Lambda^0$-valued spinorial images and the spinorial covariant gradient. The left and right Maxwell equations, (2.3) can be seen to be equivalent to

\[ \nabla_{\mu} \delta_{\mu} = 0, \quad (2.10a) \]
\[ \nabla_{\bar{\mu}} \delta_{\bar{\mu}} = 0. \quad (2.10b) \]

Now, with the II's of pure parities represented according to

\[ \Pi = 2\Pi_{AB} S_{AB}, \quad (2.11a) \]
\[ \bar{\Pi} = 2\bar{\Pi}_{AB} S_{AB}, \quad (2.11b) \]

we can compute—by applying the spinorial Ricci formulas given in Ref. 6 as (2.7a)–(2.7b)—the Laplacians and anti-Laplacians of II's. The result is

\[ \Delta \Pi = -2S_{AB} [(\Pi + R/3)\Pi_{AB} + C_{ABCD} \Pi^C D], \quad (2.12a) \]
\[ \Delta \bar{\Pi} = -2\bar{\Pi}_{AB} [(\Pi + R/3)\Pi_{AB} + C_{ABCD} \bar{\Pi}^C D], \quad (2.12b) \]

and

\[ \Delta \Pi = 2\nabla_{\mu} \nabla_{\rho} S_{\mu \rho} = \nabla_{\mu} \delta_{\mu}, \quad (2.13a) \]
\[ \Delta \bar{\Pi} = 2\nabla_{\bar{\mu}} \nabla_{\bar{\rho}} \bar{S}_{\bar{\mu} \bar{\rho}} = \nabla_{\bar{\mu}} \delta_{\bar{\mu}}, \quad (2.13b) \]

where we introduced the abbreviation

\[ \Box = -\frac{1}{2} \nabla_{\mu} \nabla^{\mu}; \quad \Lambda^2 \rightarrow \Lambda^0. \quad (2.14) \]

Consequently, Eqs. (2.10) are satisfied by

\[ f_{AB} = \frac{1}{2} \nabla_{\mu} \frac{\partial}{\partial x^A} \Pi_{\mu S} \delta_{S B}, \quad (2.15a) \]
\[ \bar{f}_{AB} = \frac{1}{2} \nabla_{\bar{\mu}} \frac{\partial}{\partial x^A} \bar{S}_{\bar{\mu} S} \delta_{S B}, \quad (2.15b) \]

provided that the $D(0,1)$ and $D(1,0)$ Hertz potentials fulfill correspondingly

\[ (\frac{1}{2} + R/3)\Pi_{AB} + C_{ABCD} \Pi^D C = 0, \quad (2.16a) \]
\[ (\frac{1}{2} + R/3)\bar{\Pi}_{AB} + C_{ABCD} \bar{\Pi}^D C = 0. \quad (2.16b) \]

Now, if the gauge forms $\nu, \bar{\nu}$ from (2.7) are represented according to

\[ \nu = 2\mu_{\mu AB} S_{AB}, \quad (2.17a) \]
\[ \bar{\nu} = 2\bar{\mu}_{\bar{\mu} AB} S_{AB}, \quad (2.17b) \]

then Eqs. (2.8) and (2.9) can be stated in the equivalent scalar form as

\[ \nabla_{\mu} \delta_{\mu} = -\frac{1}{2} \nabla_{\mu} \delta_{\mu}, \quad \Box \mu = 0. \quad (2.18a) \]

By using the freedom of $\nu$-gauges, one can bring the Hertz potentials to various plausible or useful forms. For the reduction of the last to the Debye potentials, see Ref. 1; for some applications of the Debye potentials in the theory of Einstein-Maxwell equations see Ref. 7.

In heavens (in particular, in flat space–time) there exists a "homogeneous spinor" $\mathcal{K}_4 + 0$, such that

\[ \Delta \mathcal{K}_4 = 0. \quad (2.19) \]

[Indeed, according to (1.13), the integrability condition of (2.19) is $\Delta \mathcal{K}_4 = 0 = K_{4A}$, but precisely in (strong) heavens $C_{ABCD} = C_{ABCD} = R_{ABCD} = 0$. Using then the freedom of $\nu$ gauge one can bring the left [i.e., $D(0,1)$ Hertz potential] to the particularly simple and plausible form of

\[ \Pi_{4 AB} = H \mathcal{K}_4^A \mathcal{K}_4^B \quad (2.20) \]

characterized by the algebraic condition

\[ \nabla_{\mu} \mathcal{K}_4^A = 0. \quad (2.21) \]

Equation (2.16b) reduces then to the simple scalar equation:

\[ C \mathcal{H} = 0. \quad (2.22) \]

For flat space–time, this specialization of the Hertz potential was obtained many years ago by Penrose.\(^{8}\) For the sake of completeness, we shall derive it again in the present notation. In flat space–time, there exists a frame such that

\[ \mathbf{g}^{AB} = \delta^{AB} = dx^A dx^B \quad (2.23) \]

with $X^A \in \Lambda^0$ being (Cartesian) coordinates. We describe this as a "special frame" of "s.f." The flat tetrad $\mathbf{g}^{AB}$ induces, of course connections $\Gamma^A_{BD}$ and $\bar{\Gamma}^A_{BD}$ such that

\[ \mathbf{D} \bar{\Gamma}^A_{BD} = 0 = \mathbf{D} \Gamma^A_{BD}, \quad (2.24) \]

and in our special frame $\bar{\Gamma}^A_{BD} = 0 = \Gamma^A_{BD}$. Consequently, we have

\[ \nabla_{\mu} \mathbf{A}^A = \mathbf{D} \mathcal{K}_4^A \quad (2.25) \]

in the obvious notation, so that the left Maxwell equations amount to the simple

\[ (\partial / \partial X^A) \mathcal{F}^A_C = 0, \quad (2.26) \]

in our s.f. Let now $Z_4$ be any two (out of the four) independent variables. We have then a simple lemma: The condition $\partial \mathcal{F}^A / \partial X^A = 0$ implies (and is implied by) the existence of such a $\mathcal{A}_4$ that $\mathbf{g}^{AB} = \delta^{AB} = \partial \mathbf{X}^A / \partial X^A$. Therefore, equations $\partial \mathcal{F}^A / \partial X^A = 0$ imply $f_{AB} = \partial \mathbf{A}^A / \partial X^A$, but because $\mathbf{g}^{AB}$ is symmetric, $\partial \mathbf{A}_4 / \partial X^A = 0$, and by again applying the lemma we infer the existence of such a scalar $\mathbf{H}$ that

\[ f_{AB} = \frac{1}{2} (\partial / \partial X^A) (\partial / \partial X^B) \mathbf{H}. \quad (2.27) \]

Substituting in the remaining equations $(\partial / \partial X^A) \mathbf{F}^C_A = 0$, we obtain

\[ (\partial / \partial X^A) (\Box \mathbf{H}) = 0, \quad (2.28) \]

where $\Box = -\frac{1}{2} \nabla_{\mu} \nabla^{\mu}$. Consequently, $\Box \mathbf{H} = \mathbf{A}^A (\partial / \partial X^A \mathbf{A}^A)$, where the function of the
two variables $\alpha$ is arbitrary. However, regauging $H$ according to $H \rightarrow H + \rho_\alpha (X^i) \xi^A$ we leave (2.27) unchanged, while the arbitrary $\rho_\alpha (X^i)$ can always be so chosen that the regauged $H$ fulfills already the homogenenous equation:

$$v^a H = 0.$$  \hfill (2.29)

It is now clear that by introducing a constant spinor

$$K_A = (1,0) - \bar{A} = (0,1), \quad \bar{a}K_A = 0$$  \hfill (2.30)

we can rewrite (2.27) in an arbitrary $\zeta$-frame precisely in the form (2.20).

Therefore, extending the argument given above on the right side and summing up, we can state the following: In flat (possibly complex) space-time, the Hertz potentials for the most general (vacuum) electromagnetic field can be always so gauged that

$$\Pi_{\hat{A}A} = HR_{\hat{A}A}, \quad \bar{a}K_{\hat{A}A} = 0, \quad v^A H = 0,$$  \hfill (2.31a)

$$\Pi_{AA} = HR_{AA}, \quad \bar{a}K_{AA} = 0, \quad v^A H = 0, \quad (2.31b)$$

with the homogenenous spinors $K_{\hat{A}} = 0 \neq K_A$ being otherwise arbitrary. The spinorial forms of the electromagnetic field are then given by (2.15) [with $\nabla_{\hat{A}A}$ replaced by commuting $\nabla_{A\hat{A}}$], and, the final general solution of the Maxwell equations amounts to

$$\omega = -d\alpha, \quad \bar{\omega} = :g^A \nabla_{\hat{A}} (\Pi_{\hat{B}}),$$  \hfill (2.32a)

$$\bar{\omega} = -d\alpha, \quad \bar{\omega} = :g^A \nabla_{A} (\Pi_{\hat{B}}).$$  \hfill (2.32b)

In the real case $\omega = (\omega)^*$ and $\bar{\omega} = (\bar{\alpha} + \bar{\alpha}) XM^*$, where the real $A_u$ are electric potentials $(f_{\mu v} \rightarrow \alpha_{\mu v} - \bar{\alpha}_{\mu v})$ and the pure imaginary $A_{\bar{u}}$ are magnetic potentials $(f_{\mu v} \rightarrow (2\sqrt{-g}) \varepsilon^{\mu v} \bar{\alpha}_{\mu v} + \bar{\alpha}_{\mu v} - \bar{\alpha}_{\mu v})$.

Therefore, the integral varieties of the left and right Maxwell equations in the flat space are entirely determined by the integral varieties of the simple equations

$$\bar{a}K_{\hat{A}} = 0 \neq H$$

and

$$\bar{a}K_{\hat{A}} = 0 \neq \bar{H}$$

If it is assumed that the key function fulfills the second heavenly equation:

$$\Theta_{\mu v} - \Theta_{\mu \nu} = \Theta_{\nu \mu} - \Theta_{\nu \mu} = 0.$$  \hfill (3.4)

The only nontrivial curvature objects which accompany this "heavenly tetrad" are the components of $C_{ABCD}$ given by

$$C_{1111} = \Theta_{\mu \nu}, \quad C_{1112} = \Theta_{\mu \nu}, \quad C_{1122} = \Theta_{\mu \nu}, \quad C_{2222} = \Theta_{\mu \nu}.$$  \hfill (3.5)

In order to provide a fully covariant description of these results, introduce in the $\zeta$-frame used above a homogeneous hellish spinor:

$$K_{\hat{A}} = (1, 0) - \bar{K} = (0, 1), \quad \bar{a}K_{\hat{A}} = 0$$  \hfill (3.6)

[see comment after (2.19)]. Then, in the same frame of the spinorial gauge, we can introduce the spinorial coordinates labelled by the two indices:

$$\{(v^A \hat{A})_{\hat{A}}, (dX^A \hat{A}) = :\sqrt{2} \left( -dv \right)$$  \hfill (3.7)

We can now introduce—all the time in the same $\zeta$-frame—a new $D(0, 2)$ object.
\[ \Pi_{4} h_{\alpha} \delta_{\beta} := 2 \Theta K_{A} K_{\alpha} K_{\beta}, \quad (3.8) \]

which, with \( \Theta \) treated as a scalar, is then defined in any \( \zeta \)-frame.

It is then a simple algebraic exercise to show that (3.1) and (3.2) can be equivalently rewritten in the form of

\[ g^{A}_{\alpha} = dX^{A}_{\alpha} - dx^{C}_{\alpha} \frac{\partial}{\partial x^{C}_{\alpha}}(g^{A}_{\beta}) \frac{\partial}{\partial x^{C}_{\beta}}(\Pi \delta^{A}_{\beta}), \quad (3.9) \]

and

\[ - \frac{1}{2} \delta^{A}_{\alpha} \delta^{B}_{\beta} = \left( \frac{\partial}{\partial x^{A}_{\alpha}} \frac{\partial}{\partial x^{B}_{\beta}} \right) \left( \frac{\partial}{\partial x^{C}_{\alpha}} \frac{\partial}{\partial x^{C}_{\beta}} \right) \Pi \delta^{A}_{\beta} \delta^{B}_{\alpha}. \quad (3.10) \]

At the same time, the formulas for the connections and the curvature assume the form:

\[ \Gamma_{AB}^C = \frac{1}{2} \left( \frac{\partial}{\partial x^{C}_{\alpha}} \frac{\partial}{\partial x^{B}_{\alpha}} \right) \left( \frac{\partial}{\partial x^{A}_{\alpha}} \frac{\partial}{\partial x^{A}_{\beta}} \right) (\Pi \delta^{C}_{\alpha} \delta^{C}_{\beta}), \]

\[ \Gamma_{AB}^C = 0, \quad (3.11) \]

and

\[ \frac{1}{2} C_{ABCD} = \frac{1}{2} \left( \frac{\partial}{\partial x^{A}_{\alpha}} \frac{\partial}{\partial x^{B}_{\alpha}} \right) \left( \frac{\partial}{\partial x^{C}_{\alpha}} \frac{\partial}{\partial x^{D}_{\alpha}} \right) (\Pi \delta^{A}_{\beta} \delta^{B}_{\gamma} \delta^{C}_{\delta} \delta^{D}_{\epsilon}). \quad (3.12) \]

The second heavenly equation can be then expressed in the terms of the object (3.5):

\[ 4(\Theta_{
abla} + \Theta_{e} + \Theta_{\nabla} \Theta_{\nabla} - \Theta_{\nabla} \Theta_{\nabla} K_{A} K_{\alpha} K_{\beta}), \quad (3.13) \]

where, of course, we denoted

\[ \Theta_{\nabla} := \frac{1}{2} \Theta_{\nabla} \delta_{\alpha} \delta_{\beta}, \quad (3.14) \]

It follows that we can now write our formulas which describe \( \hat{H} \)-space in an arbitrary \( \zeta \)-frame in the simple form of

\[ S'_{AB} = \gamma_{AB} + \frac{1}{2} \delta_{A} \delta_{B} \Theta_{\nabla} \Theta_{\nabla} C_{A} C_{B} \Pi \delta^{A}_{\beta} \delta^{B}_{\alpha}, \quad (3.15) \]

\[ \Theta_{AB} = \gamma_{AB} - \frac{1}{2} \Theta_{A} \Theta_{B} \Theta_{\nabla} \Theta_{\nabla} C_{A} C_{B} \Pi \delta^{A}_{\beta} \delta^{B}_{\alpha}, \quad (3.16) \]

\[ \Theta_{\nabla} = \Theta_{\nabla} \Theta_{\nabla} K_{A} K_{\alpha} K_{\beta}, \quad (3.17) \]

\[ H_{\nabla} \Theta_{AB} = \gamma_{AB}, \quad (3.18) \]

\[ C_{ABCD} = \frac{1}{2} \Theta_{A} \Theta_{B} \Theta_{C} \Theta_{D} \Pi \delta^{A}_{\beta} \delta^{B}_{\alpha} \delta^{C}_{\delta} \delta^{D}_{\epsilon}, \quad C_{ABCD} = 0, \quad (3.19) \]

\[ \Theta_{AB} = \Theta_{AB} \Theta_{\nabla} K_{A} K_{\alpha} K_{\beta}, \quad (3.20) \]

\[ c^{c} \Pi \delta^{A}_{\beta} \delta^{B}_{\alpha} \delta^{C}_{\delta} \delta^{D}_{\epsilon}, \quad (3.21) \]

where, of course, we denoted

\[ c^{c} = \frac{1}{2} \Theta_{A} \Theta_{B} \Theta_{C} \Theta_{D} \Pi \delta^{A}_{\beta} \delta^{B}_{\alpha} \delta^{C}_{\delta} \delta^{D}_{\epsilon}, \quad (3.22) \]

\[ \tilde{D} = - \iota_{D} \Theta_{\nabla}, \quad (3.23) \]

that is, the "covariant" Laplacian \( \tilde{D} \tilde{D} + \tilde{D} \tilde{D} \): \( \hat{D} \rightarrow \tilde{D} \rightarrow \hat{D} \) coincides in \( \hat{D} \) with operation \( \iota_{D} \); of course working with \( \tilde{D} \) we obtain then \( \iota_{D} \).

We can add that by using the results of Ref. 2 concerning the \( S \) objects in our s.f., and then by generalizing them to an arbitrary \( \zeta \)-frame, we can accompany formulas (3.15)–(3.21) by

\[ \begin{align*}
\Theta_{AB} & = \gamma_{AB} - \frac{1}{2} \Theta_{\nabla} \Theta_{\nabla} K_{A} K_{\alpha} K_{\beta}, \\
\Theta_{\nabla} & = \Theta_{\nabla} \Theta_{\nabla} K_{A} K_{\alpha} K_{\beta}, \\
\Theta_{\nabla} & = \Theta_{\nabla} \Theta_{\nabla} K_{A} K_{\alpha} K_{\beta}.
\end{align*} \quad (3.22) \]

To close the covariant description of heavens given by these formulas, we must, of course, list the information that

\[ \Theta_{AB} = 0, \quad \Theta_{\nabla} \Theta_{\nabla} K_{A} K_{\alpha} K_{\beta} = 0, \] defining a homogenous spinor. \quad (3.24)
\( \Gamma_{\alpha \beta} \equiv \frac{1}{2} g^{cd} \left( \partial / \partial x^a \partial / \partial x^b \right) (\partial / \partial x^c \partial / \partial x^d) R_{\beta \alpha}^{\gamma \delta} \). \hspace{1cm} (3.31)

Applying then the rule
\[ -\Gamma_{\alpha \beta} u^\alpha \Gamma_{\beta \gamma} = \partial_\gamma c_\beta \beta \]  \hspace{1cm} (3.32)

[where, of course, \( \gamma_{\alpha \beta} = : g_{\alpha \beta} \partial \text{d} x^\nu \text{ and local-coordinate indices are manipulated by the Riemannian metric } g_{\nu \nu} \)], we easily infer that
\[ -\frac{1}{2} \Gamma_{\alpha \beta} u^\alpha c_\beta \beta = (\partial / \partial x^\beta / \partial x^\gamma ) (\partial / \partial x^\delta ) (\partial / \partial x^\gamma ) R_{\beta \alpha}^{\gamma \delta} \]  \hspace{1cm} (3.33)

We can now prove a lemma that
\[ \nabla_{A_{ij}} \cdots \nabla_{A_{ij}} u^i \cdots u^i c_{ij} \cdots c_{ij} = 0 \]  \hspace{1cm} (3.34)

for \( k = 1, 2, 3, 4 \). We prove this by induction with respect to \( k \).

For \( k = 1 \), because of (3.29) we have
\[ \nabla_{A_{ij}} \cdots \nabla_{A_{ij}} u^i = c_{ij} \]  \hspace{1cm} (3.35)

We assume now (3.34) for some \( k > 1 \). Then
\[ \nabla_{A_{ij}} \cdots \nabla_{A_{ij}} u^i = c_{ij} \]  \hspace{1cm} (3.36)

where \( \cdots \) denotes the terms with \( \Gamma^i_{\alpha \beta} u^\alpha \beta \) which correspondingly take care of all undotted indices. Because, however, of (3.33), all these terms contain the contraction \( K_{ij} \Pi_{\gamma \delta} \cdots \), and hence all vanish. Using thus in the term with \( c_{ij} \) the second of the properties (3.29), we can replace it by \( c_{ij} \). We have therefore
\[ \nabla_{A_{ij}} \cdots \nabla_{A_{ij}} u^i = c_{ij} \]  \hspace{1cm} (3.37)

Being valid in an s.f., this covariant equation is valid in any frame. This concludes the inductive proof of (3.34).

We still must demonstrate—as the last lemma necessary for our purposes—that
\[ \square \Pi_{A_{ij} \hat{A}_{ij}} = \square \Pi_{A_{ij} \hat{A}_{ij}}. \]  \hspace{1cm} (3.38)

We again prove it in our s.f. First, we have
\[ \square \Pi_{A_{ij} \hat{A}_{ij}} = -\frac{1}{2} \nabla_{B \hat{B}} \nabla_{B \hat{B}} \Pi_{A_{ij} \hat{A}_{ij}} = -\frac{1}{2} \nabla_{B \hat{B}} \nabla_{B \hat{B}} \Pi_{A_{ij} \hat{A}_{ij}} \]  \hspace{1cm} (3.39)

where for \( \partial_\gamma \delta \) we can use (3.10), and in the last term we can apply (3.33). In doing so, it is relevant to remember a formal property of the tangent spinors
\[ (\partial / \partial x^\gamma ) \hat{R}_{\alpha \beta} = -(\partial / \partial x^\beta ) \hat{R}_{\alpha \beta}, \]  \hspace{1cm} (3.40)

which was first clearly encountered and discussed in Ref. 4 [this property is important also when rewriting (3.9) and (3.10) in the covariant form of (3.15) and (3.16); (3.40) explains the necessity of an additional change of sign in these formulas]. We obtain, therefore,
\[ \square \Pi_{A_{ij} \hat{A}_{ij}} = -2[(\partial / \partial x^\gamma ) \hat{R}_{\alpha \beta}] (\partial / \partial x^\beta ) \hat{R}_{\alpha \beta} \]  \hspace{1cm} (3.41)

Executing here the differentiations, cancelling and dropping all terms which contain the contractions like \( K_{ij} \hat{R}_{ij} \), we easily find that
\[ \square \Pi_{A_{ij} \hat{A}_{ij}} = -2(\partial / \partial x^\gamma ) \hat{R}_{\alpha \beta} \Pi_{A_{ij} \hat{A}_{ij}} = -\frac{1}{2} \nabla_{B \hat{B}} \nabla_{B \hat{B}} \Pi_{A_{ij} \hat{A}_{ij}}. \]  \hspace{1cm} (3.42)

Valid in an s.f., this covariant equation must also hold in any \( \mathcal{C} \)-frame, and thus (3.88) is proven. [Notice that already in Ref. 2 treating \( \Theta \) as a scalar it was established that \( \Theta = \Theta \mathcal{G} \mathcal{G} \) which, with a constant \( K_{ij} \), is essentially equivalent to (3.38); the present proof is given in order to assure the completeness of this paper.]

We shall see that the sequence of the lemmas demonstrated above proves our theorem; indeed, that we can replace \( g_{\alpha \beta} \) by \( g_{\alpha \beta} \) in all formulas beginning from (3.15) to (3.23) follows from the lemma (3.34); then, (3.36) guarantees that we can replace \( \hat{R}_{ij} \) by \( \hat{R}_{ij} \) when acting on \( \Pi_{A_{ij} \hat{A}_{ij}} \). As far as the operation \( \hat{D} \) is concerned, we have already (3.28): \( \hat{D} \in (3.23) \) amounts to the operation on the object with pure dotted indices but that with \( \Gamma_{ij} = \hat{T}_{ij} \) it can be again replaced by \( D \). Eventually, the fact that we can replace the terms with \( \Pi_{A_{ij} \hat{A}_{ij}} \) the \( S \)'s by \( \hat{S} \)'s in (3.22) and (3.23) is the consequence of the fact \( S_{ij} = \hat{S}_{ij} \) (adding something) \( K_{ij} \hat{K}_{ij} \) and that \( \hat{D} \) is a scalar operator referring to \( g_{\alpha \beta} \) by those referring to \( g_{\alpha \beta} \), or completely, eliminating the objects and operators referring to \( g_{\alpha \beta} \) by those referring to \( g_{\alpha \beta} \) according to the scheme (3.27) in the formulas (3.15)–(3.16)—of course, only in the terms involving \( \Pi_{A_{ij} \hat{A}_{ij}} \) or \( K_{ij} \)—is therefore leading to valid formulas either if we execute these replacements only partially, i.e., in some of these formulas, or completely, eliminating the objects and operators referring to \( g_{\alpha \beta} \) in all possible places. In the latter case, we obtain therefore a set of formulae...
In the terms of the gravitational Hertz potential, the various things is selected to be of the type N, with the quadruple
commute on
and, more generally, of the double K–S metrics (see Ref. 14 and then Ref. 15 for the general theory of the last metrics).

Suppose, for example, that
where \( \gamma \) is flat and \( K \in A^1 \) is null (i.e., \( KJK = 0 \)) and geodesic with respect to \( \gamma \) or \( g \). It is then null and geodesic with respect to the both, and its optical scalars are the properties underlying our theorem.

4. COMPLEX GRAVITY IN LINEAR APPROXIMATION

The dynamical equations of general relativity have been studied in the linear approximation from many points of view. Our results on \( \mathcal{H} \)-spaces suggest another approach. \( \mathcal{H} \)-space with the tetrad oriented as in the previous section, provides the most general right-flat solution of Einstein’s equations. Changing the orientation (by making a tetrad transformation of determinant minus one), we obtain the most general left-flat solution. From these solutions of the rigorous equations, we derive solutions to the equations of the linear approximation. For these equations, however, we can superimpose solutions. By this means, we recover the general solution of the linear approximation which Penrose obtained from completely different considerations.

Following the program outlined above, we first construct the general “left-flat” solution. The null tetrad transformation \( c^1 \to c^2, c^2 \to c^1, c^1 \to c^2 \) corresponds to the replacement of the dotted indices by undotted and vice-versa, i.e., we obtain the helless tetrad from the formulas (3.15)–(3.26) formally by “conjugating” and treating the objects \( \gamma_{\mathcal{H}}^{AB}, \gamma^*_{\mathcal{H}}^{AB}, \gamma^{\mathcal{H}}_{\mathcal{H}} \) (and \( \mathcal{S}_0 \)) as if they were “Hermitian”. This leads to the following list of formulas:

\[
\begin{align*}
\gamma^*_{\mathcal{H}}^{AB} & = \gamma^*_{\mathcal{H}}^{AB} - i \gamma^*_{\mathcal{H}}^{CD} \gamma^*_{\mathcal{H}}^{BC} (\gamma^{\mathcal{H}}_{\mathcal{H}})^{-1} (\gamma^{\mathcal{H}}_{\mathcal{H}})^{-1}, \\
\Gamma_{\mathcal{H}}^{AB} & = \Gamma_{\mathcal{H}}^{AB}, \\
\Gamma_{\mathcal{H}}^{CD} & = \Gamma_{\mathcal{H}}^{CD}, \\
\mathcal{A}_{\mathcal{H}} & = 0,
\end{align*}
\]

In the terms of the gravitational Hertz potential, the gravitational field (of helicity \( \pm 2h \)) \( C_{\mathcal{ASC}} \) is given by

\[
\begin{align*}
\gamma^*_{\mathcal{H}}^{AB} & = \gamma^*_{\mathcal{H}}^{AB} - i \gamma^*_{\mathcal{H}}^{CD} \gamma^*_{\mathcal{H}}^{BC} (\gamma^{\mathcal{H}}_{\mathcal{H}})^{-1} (\gamma^{\mathcal{H}}_{\mathcal{H}})^{-1}, \\
\Gamma_{\mathcal{H}}^{AB} & = \Gamma_{\mathcal{H}}^{AB}, \\
\Gamma_{\mathcal{H}}^{CD} & = \Gamma_{\mathcal{H}}^{CD}, \\
\mathcal{A}_{\mathcal{H}} & = 0,
\end{align*}
\]

We should like to emphasize the striking analogy of this result with the description of the general left and right electromagnetic fields in flat space-time in the terms of the Hertz potentials in the null gauge, formulas (2.31) and (2.32). We also notice the analogy of our result concerning the theorem about the possibility of equivalently replacing objects and operators referred to \( \gamma^*_{\mathcal{H}}^{AB} \) and \( \gamma^{\mathcal{H}}_{\mathcal{H}} \), with similar mechanisms which we encounter in the theory of the Kerr–Schild metrics.
It is self-evident that our theorem on the possibility of replacing objects and operators according to the scheme (3, 27) in all terms containing $\Pi_{ABC}$ (or $K_{A}$) applies also the present collection of the formulae. Clearly, $\Pi_{ABC}$ plays the role of the right (null) gravitational Hertz potential for the space $H$.

In the next step, we consider, in both sets of the formulas (3, 15)-(3, 26) $[H]$ and (4, 1)-(4, 12) $[H]$, the corresponding Hertz potentials as the quantities of the first order, $\Pi_{ABC}$, $\Pi_{ABCD}$, $\delta \Pi_{ABC}$ (i.e., $\Theta \rightarrow \Theta$, $\Theta \rightarrow \Theta_{0}$), where $\Theta_{0}$ denotes the order in the parameter of smallness. Then, neglecting the terms of higher order and superposing linearly both structures, we obtain for the solutions of Einstein equations which are both-sidedly general, but only infinitesimally deviate from the flatness ($\delta G \odot G$ solutions) the following collection of formulas:

\[
\delta_{AB} = \delta_{AB} + \frac{1}{4} \epsilon_{ABCD} \nabla_{C} \nabla_{D} \delta_{0} (\Pi_{ABCD} - \Pi_{ABCD}),
\]

\[
\delta_{AB} = \delta_{AB} + \frac{1}{4} \epsilon_{ABCD} \nabla_{C} \nabla_{D} \delta_{0} (\Pi_{ABCD} - \Pi_{ABCD}),
\]

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5. FINAL REMARKS

The fundamental question arises concerning how our $\delta G \odot G$ structure generalizes within the complete nonlinear theory. For the (complex) space-times which are one-sidedly flat, i.e., heavens $[-] \odot G$ and $G \odot [-]$ we know the answer: One of (infinitesimal) Hertz potentials of the linear approximation goes to the zero limit, while the second potential becomes finite and fulfills a simple nonlinear equation with quadratic non-linearity, maintaining from the linear approximation two crucial properties (i) its type N, (ii) the proportionality of the quadruple Penrose spinor to a homogeneous spinor.

In the general case of the solutions of the Einstein equations of the type $G \odot G$, the present results seem to suggest strongly that it should be possible to describe entirely these solutions in the terms of some two Hertz potentials of the types $D(0,2)$ and $D(2,0)$ respectively. How this should be done can perhaps become more transparent when the basic results of this paper concerning the “spinorization” and “covariantization” of the $H$-spaces, will be extended to the theory of $\mathcal{H}/H$-spaces.

The last spaces, being the solutions of Einstein (empty space) equations of the type $\mathcal{D} \odot \mathcal{G}$, are entirely described in the terms of one function of four variables and some (gauge dependent) functions of the two variables. (See Refs. 9 and 18 for succinct resumé, and 17 for the complete proofs; Ref. 18 contains a spinorial description of $D \odot G$ spaces but with the SL gauge completely “frozen” from the $D$ side and partially restricted from the $G$ side; Refs. 19 and 20 contain the generalization of the theory of $\mathcal{H}/H$ spaces on the case of Einstein–Maxwell equations and then subsequent spinorial description of the results obtained; Ref. 21 contains comparison of the results of the theory of the type D solutions as stated in Ref. 22 with the theory of $\mathcal{H}/H$ spaces). The $\mathcal{H}/H$ equation—very similar to the second heavenly equation (3.4)—is the only condition on the function of four variables which

\[
\delta_{0} \Theta = 0, \quad \delta_{0} \Theta = 0.
\]

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becoming this way very similar to Eqs. (2.3) for the present time whether Eqs. (5.1) can be directly approached as the starting point in introducing the corresponding gravitational Hertz potentials. We also consider at the present time whether Eqs. (5.1) can be directly approached as the starting point in introducing the corresponding gravitational Hertz potentials. We also believe that some relevant hints concerning the structure of the \( G \otimes G \) solutions described by Hertz potentials can be obtained by using \( 5G \otimes 5G \) structure as the first order approximation in a covariant approximation procedure which would permit us to determine all pertinent quantities with the precision up to one order higher [up to \( O(\delta) \)]. Some work in this direction (jointly with Dr. S. Hacyan) is now in progress.

It can be also noticed, that the structure equations with the built-in Einstein equations \( C_{ABCD} = 0 \), \( R = -4\lambda \) can be stated together with Bianchi identities in the form of

\[
\begin{align*}
*R_{AB} &= R_{AB}, \quad DR_{AB} = 0, \quad \bar{D}R_{AB} = 0, \quad (5.1a) \\
*R_{\hat{A}\hat{B}} &= -R_{\hat{A}\hat{B}}, \quad DR_{\hat{A}\hat{B}} = 0, \quad DR^*_{\hat{A}\hat{B}} = 0, \quad (5.1b)
\end{align*}
\]

(where \( \bar{D} = -i^*D^* \) is the covariant codifferential), becoming this way very similar to Eqs. (2.3) for the Maxwellian field, which suggest the usefulness of the electromagnetic Hertz potentials. We also consider at the present time whether Eqs. (5.1) can be directly approached as the starting point in introducing the corresponding gravitational Hertz potentials.

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