Deformations of algebraic types of the energymomentum tensor
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Arnold's minimal deformations of the matrix canonical forms for the traceless Ricci tensor $\mathcal{C}$ are found. The diagram presenting all possible degenerations for the algebraic types of $\mathcal{C}$ is given.

I. INTRODUCTION

Algebraic classification of the energy-momentum tensor $T_{\mu\nu}$, $\mu, \nu = 1, ..., 4$, with the use of matrix algebra has been carried out by Churchill, Petrov, Plebański, and Hall. In Refs. 4, 5 a spinor approach to this classification is also proposed. A general spinor technique has been presented and applied to the classification of $T_{\mu\nu}$ by Penrose.

In the present paper, we follow Refs. 4 and 5. According to the results of these articles there exist 15 algebraic types of $T_{\mu\nu}$, which are divided into four Types (capital T): $T_1$, $T_2$, $T_3$, and $T_4$. $T_{\mu\nu}$ is of Type $T_i$ if $T_{\mu\nu}$ has four eigenvectors and some of its eigenvalues are complex. If $T_{\mu\nu}$ has four eigenvectors and all its eigenvalues are real then $T_{\mu\nu}$ belongs to type $T_i$. Finally, if $T_{\mu\nu}$ has three or two eigenvectors then $T_{\mu\nu}$ is of type $T_3$ or $T_4$, respectively (compare Fig. 1 in Sec. III of the present paper). Now it seems to be physically reasonable to impose some restrictions on the energy-momentum tensor representing real macroscopic matter. These restrictions are known as the dominant energy condition, defined by

$$T_{\mu\nu}u^\mu u^\nu > 0 \quad (1.1a)$$

and

$$T_{\mu\nu}v^\nu > 0$$

(1.1b)

for every non-spacelike vector $u^\mu$. The condition (1.1a) is called the weak energy condition. In the theory of singularities in a space-time the following conditions are imposed on $T_{\mu\nu}$.

(i) The null convergence condition

$$T_{\mu\nu}u^\mu u^\nu > 0 \quad (1.2)$$

for every null vector $u^\mu$.

(ii) The timelike convergence condition

$$T_{\mu\nu}v^\mu v^\nu \geq (1/8\pi) \Lambda$$

(1.3)

for every non-spacelike vector $v^\nu$ ($\Lambda$ is the cosmological constant). The following relation holds:

$$T_{\mu\nu}u^\mu u^\nu \Rightarrow (1.1a) \Rightarrow (1.2) \Rightarrow (1.3). \quad (1.4)$$

As has been shown in Refs. 4 and 9, neither (1.1a) nor (1.1b) can be satisfied for $T_{\mu\nu}$ of type $T_2$ or $T_3$. Therefore, these types are physically unrealized, at least on the macroscopic level. Similarly, the condition (1.2) and, by (1.4), also the condition (1.3) cannot be satisfied by $T_{\mu\nu}$ of type $T_3$ or $T_4$. We consider these facts further in Sec. IV.

There are various interesting applications of the algebraic classification of $T_{\mu\nu}$. Some of them can be found in Refs. 4 and 9-15.

The natural question now arises: Is the algebraic classification of $T_{\mu\nu}$ stable under small perturbations? This question is natural and important from the physical point of view, as no physical measurement determining the energy-momentum tensor can be exact and, consequently, the components of this tensor are known only to some finite accuracy.

An elegant and effective method which enables us to answer the question stated has been developed by Arnold, who considers the normal forms for matrices depending holomorphically on parameters. Arnold's theory has been applied by Ellis and McCarthy for finding the stable canonical forms of the Weyl tensor.

In the present paper we follow Refs. 16 and 17 with some obvious modifications (see Secs. II and III).

The main results are presented in Sec. III, where we find the minimal deformations of all 15 canonical forms for the traceless Ricci tensor and then get the stable classification of the energy-momentum tensor [see (3.39)]. The scheme (3.39) yields also all possible degenerations for the algebraic types of $T_{\mu\nu}$. (Fig. 1). Figure 1 generalizes some results of Refs. 4 and 5 (compare Table II of Ref. 4). Some results of Sec. III can be found in Ref. 18.

From the physical point of view, the most important conclusion of our results can be derived from (3.39) or Fig. 1. It states that any algebraic type of $T_{\mu\nu}$ except $[3S - T]_2$, $[S_1 - S_2 - S_3 - T]$ is deformable by a small perturbation to "unphysical" types $T_i$, and/or $T_3$. Consequently, for all types of $T_{\mu\nu}$ except possibly $[3S - T]_2$, $[S_1 - S_2 - S_3 - T]$, and $[S_1 - S_2 - S_3 - T]$ any of the energy conditions (1.1a) and (1.2), (1.3) appear to be unstable under small perturbations. We consider this fact in Sec. IV.

II. ARNOLD'S THEORY OF DEFORMATION

Let $X$ be a finite-dimensional differentiable manifold, and let $G$ be a Lie group acting on $X$ on the left, $G \times X \ni (g,x) \mapsto g.x \in X$. Moreover, let $U$ be an open neighborhood of 0 in $\mathbb{R}^n$ (if $X$ is a real manifold) or $\mathbb{C}^n$ (if $X$ is a complex manifold), where $\mathbb{R}^n$ and $\mathbb{C}^n$ denote vector spaces of
Type I, 4 eigenvectors; not all eigenvalues real
Type II, 3 eigenvectors
Type III, 2 eigenvectors

FIG. 1. Degeneration scheme for the algebraic types of \( \mathcal{C} \).

n-tuples of real and complex numbers, respectively.

A deformation of \( x_0 \in X \) is a differentiable (respectively, holomorphic) mapping \( U \ni \lambda \rightarrow x(\lambda) \in X \) such that \( x(0) = x_0 \). Analogously one defines a deformation of an element \( g_0 \in G \). Two deformations \( x(\lambda) \) and \( x'(\lambda) (\lambda \in U) \) of \( x_0 \in X \) are said to be equivalent if there exists a deformation \( e(\lambda) \) of the identity element \( e_0 \) of \( G \) such that \( x'(\lambda) = e(\lambda) \circ x(\lambda) \). A deformation \( U \ni \lambda \rightarrow x(\lambda) \in X \) of \( x_0 \in X \) is called versal if for every deformation \( U' \ni \mu \rightarrow x'(\mu) \in X \) of \( x_0 \), there exists an open subset \( U'' \) of \( U' (0 \in U'') \), a deformation \( U'' \ni \mu \rightarrow e(\mu) \in G \) of the identity element \( e_0 \) of \( G \) and a differentiable (respectively, holomorphic) mapping \( \phi: U'' \rightarrow U \) such that \( \phi(0) = 0 \) and

\[
x'(\mu) = e(\mu) \circ x(\phi(\mu)) \quad \text{for each } \mu \in U'',
\]

i.e., the deformations \( U'' \ni \mu \rightarrow x'(\mu) \in X \) and \( U'' \ni \mu \rightarrow x(\phi(\mu)) \in X \) of \( x_0 \in X \) are equivalent.

The fundamental result that enables one to find the versal deformation is the following lemma given by Arnold\(^{16}\) (see also Ref. 17).

**Lemma:** A deformation \( x(\lambda) \) of \( x_0 \in X \) is versal iff the tangent space to the family \( x(\lambda) \) at \( x_0 \) is complementary to the tangent space to the orbit \( G \cdot x_0 \) of \( x_0 \) at \( x_0 \).

It is evident that we are mostly interested in such a versal deformation of \( x_0 \in X \) which depends on the minimal number of parameters, say \( m \). Such a versal deformation is called miniversal and from Arnold's lemma it follows that

\[
m = \dim(X) - \dim(G \cdot x_0).
\]

In the next section we find the miniversal deformations of the matrix canonical forms for the traceless Ricci tensor.

III. MINIVERSAL DEFORMATIONS OF THE MATRIX CANONICAL FORMS FOR THE TRACELESS RICCI TENSOR

Let \( p \) be a point of a space-time manifold \( M \) and let \( g_{\mu \nu} \) and \( R_{\mu \nu} \) (\( \mu, \nu = 1, \ldots, 4 \)) be the components of the metric tensor at \( p \) and the components of the Ricci tensor at \( p \), respectively, with respect to some local coordinate system \( x^\mu \) [the signature of the metric is \( (+ + - -) \)]. Finally, let \( R \) denote the scalar curvature at \( p \). Then the traceless Ricci tensor \( \mathcal{C} \) at \( p \), is defined as follows:

\[
\mathcal{C} = C_{\mu \nu} \mathrm{d}x^\mu \otimes \mathrm{d}x^\nu, \quad C_{\mu \nu} = R_{\mu \nu} - \frac{1}{2} R g_{\mu \nu}.
\]

One finds that

\[
C_{\mu \nu} = C_{\nu \mu} \quad \text{and} \quad C^\nu_\mu = 0.
\]

Define

\[
\mathcal{C}' = C_{\nu}^\mu \frac{\partial}{\partial x^\nu} \otimes \mathrm{d}x^\nu,
\]

and consider \( \mathcal{C}' \) as an endomorphism \( T_p(M) \rightarrow T_p(M) \), where \( T_p(M) \) is the tangent space of \( M \) at \( p \). This enables us to classify algebraically \( \mathcal{C}' \) and, consequently, one gets the algebraic classification of \( \mathcal{C} \) which, by the Einstein's equations, is also the algebraic classification of the energy-momentum tensor. Such a classification was done by one of us (Plebański, \(^{4,5}\) see also Refs. 13-15). In the present paper we cite only the final results.

Denoting by \( (E_1, E_2, E_3, E_4) \), \( E_\mu \in T_p(M), a = 1, \ldots, 4 \), an appropriate orthonormal tetrad at \( p \) \[ \{g_{\mu \nu}, E_\mu E_\nu = g_{\mu \nu}; a, b = 1, \ldots, 4; \quad (g_{ab}) = \text{diag}(1, 1, 1, -1) \] and by \( (E_1^*, E_2^*, E_3^*, E_4^*) \), \( E_\mu^* \in T^*_p(M) \), the dual tetrad to \( (E_1, E_2, E_3, E_4) \), one has the following algebraic classification of \( \mathcal{C}' \):

1) type I

\[
\mathcal{C} = C_{ab} E^a \otimes E^b,
\]

\[
(C_{ab}) = \begin{pmatrix}
S_1 & 0 & 0 & 0 \\
0 & S_2 & 0 & 0 \\
0 & 0 & \text{Re } Z & \text{Im } Z \\
0 & 0 & \text{Im } Z & -\text{Re } Z
\end{pmatrix}
\]

\[
\text{Im } Z \neq 0, \quad S_1 + S_2 + 2 \text{ Re } Z = 0.
\]

Eigenvalues of \( \mathcal{C}' \): \( S_1, S_2, Z, \bar{Z} \).
From the work of Arnold\(^{16}\) on deformations of Jordan canonical forms of matrices and from the work of Ellis and McCarthy\(^{17}\) on deformations of Petrov types one can expect that the above given algebraic classification of the traceless Ricci tensor is highly unstable in the sense that any small perturbation may drastically change the algebraic type. To find a stable classification of \(\gamma\) we employ Arnold’s theory of deformation. Now, the differentiable manifold \(X\) (see Sec. II) is the space of all \(4 \times 4\) real symmetric matrices such that \(\begin{vmatrix} \mathbf{A} \end{vmatrix} = \begin{vmatrix} \mathbf{A}_{11} + \mathbf{A}_{22} + \mathbf{A}_{33} - \mathbf{A}_{44} \end{vmatrix} = 0\). The Lie group \(G\) acting on \(X\) is the proper and orthochronous Lorentz group \(SO'(3,1;\mathbb{R})\). \(SO'(3,1;\mathbb{R})\) acts on \(X\) as follows:

\[
(L,A) \mapsto L\mathbf{A} = \mathbf{L}'\mathbf{A} = \mathbf{L}\mathbf{A}' \quad \text{for} \quad (L\mathbf{A}) \in SO'(3,1;\mathbb{R}) \times X,
\]

\[
L \in X,
\]

where \(L\) is the transposed matrix of \(L\). Denoting by \(C_0 \in X\) the canonical form of \(\gamma\) we consider the miniversal deformation of \(C_0\). To this end we note that the tangent space of \(X\) at \(C_0\) can be identified with \(X\) considered as a vector space over \(\mathbb{R}\). Then, with this identification, every vector tangent to the orbit \(SO'(3,1;\mathbb{R})C_0\) of \(C_0\) at \(C_0\) is of the form

\[
\mathcal{L} C_0 + (\mathcal{L} C_0)\big|_0, \quad \mathcal{L} \in \mathfrak{so}(3,1;\mathbb{R}),
\]

where \(\mathfrak{so}(3,1;\mathbb{R})\) is the Lie algebra of \(SO'(3,1;\mathbb{R})\) consisting of all \(4 \times 4\) real matrices of the general form

\[
\mathcal{L} = \begin{pmatrix}
0 & l_1 & l_2 & l_3 \\
-l_1 & 0 & l_4 & l_5 \\
l_2 & l_4 & 0 & 0 \\
l_3 & l_5 & l_6 & 0
\end{pmatrix},
\]

\[
\mathcal{L} \in \mathfrak{so}(3,1;\mathbb{R}), \quad l_1, \ldots, l_6 \text{ real numbers}.
\]

Consequently, from Arnold’s lemma it follows that in order to obtain a miniversal deformation of \(C_0\) one need only find any complement in \(T_{C_0}(X) = X\) to the orbit of \(C_0\) defined by (3.9). (The miniversal deformation of \(C_0\) depends on the complement considered!) As the work of Arnold\(^{16}\) and Ellis and McCarthy\(^{17}\) shows, it is convenient to use the orthogonal complement with respect to the scalar product \(\langle \cdot, \cdot \rangle : T_{C_0}(X) \times T_{C_0}(X) \rightarrow \mathbb{R}\) defined by

\[
\langle A, B \rangle = \text{Tr}(AB).
\]

Then, according to (2.2), the number \(m\) of miniversal parameters is

\[
m = \dim(X) - \dim(SO'(3,1;\mathbb{R}) \cdot C_0)
\]

\[
= 9 - \dim(SO'(3,1;\mathbb{R}) \cdot C_0).
\]

With this in hand we find miniversal deformations of canonical forms (3.4)–(3.7)\(^{1}\), for all algebraic types of \(\gamma\).

Type I\(_3\): Here \(C_0\) is given by (3.4). Then, using (3.9) and (3.10) one finds the general form of the vector tangent to the orbit of \(C_0\) at \(C_0\) to be...
Consequently, for the type \([S_1 - S_2 - Z - \bar{Z}]\), \(\dim(SO'(3,1;\mathbb{R}) \cdot C_0) = 6\), and the tangent space of the orbit of \(C_0\) at \(C_0\) is spanned by the following vectors:

\[
C_1 := \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad C_2 := \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad C_3 := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad C_4 := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.
\]

According to (3.12) the number \(m\) of miniversal parameters is now \(m = 9 - 6 - 3\). Using the definition (3.11) one finds that any vector orthogonal to each of the vectors (3.14) takes the form:

\[
\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4 \in \mathbb{R}, \quad \epsilon_1 + \epsilon_2 + 2\epsilon_4 = 0.
\]

Therefore, we conclude that \(\epsilon_1, \epsilon_2, \epsilon_3\) are the miniversal parameters and a family \([S_1 - S_2 - Z - \bar{Z}]_4\) consisting of all matrices determined by appropriate miniversal deformations of canonical matrices for the type \([S_1 - S_2 - Z - \bar{Z}]_4\) is given by

\[
\left[ \begin{array}{cccc} S & 0 & 0 & 0 \\ 0 & S & 0 & 0 \\ 0 & 0 & \text{Re}Z & \text{Im}Z \\ 0 & 0 & \text{Im}Z & -\text{Re}Z \end{array} \right] + \left[ \begin{array}{cccc} \epsilon_1 & \epsilon_2 & 0 & 0 \\ \epsilon_2 & \epsilon_3 & 0 & 0 \\ 0 & 0 & \epsilon_5 & \epsilon_4 \\ 0 & 0 & \epsilon_4 & -\epsilon_5 \end{array} \right],
\]

\[\text{Im}Z \neq 0, \quad S + \text{Re}Z = 0; \quad \epsilon_1 + \epsilon_2 + 2\epsilon_4 = 0.\]

Consider now the type \([2S - Z - Z]_3\). From (3.13) one infers that in this case \((S_1 = S_2) \dim(SO'(3,1;\mathbb{R}) \cdot C_0) = 5\) and the orbit tangent space at \(C_0\) is spanned by the vectors \(C_2, C_3, C_4, C_5,\) and \(C_6\) defined by (3.14). The number \(m\) of miniversal parameters is \(m = 9 - 5 = 4\). Any vector orthogonal to the orbit of \(C_0\) at \(C_0\) is of the form:

\[
\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4 \in \mathbb{R}, \quad \epsilon_1 + \epsilon_2 + 2\epsilon_4 = 0.
\]

Thus \(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4\) are the miniversal parameters and the family \([2S - Z - Z]_3\) is given by

\[
\left[ \begin{array}{cccc} S & 0 & 0 & 0 \\ 0 & S & 0 & 0 \\ 0 & 0 & \text{Re}Z & \text{Im}Z \\ 0 & 0 & \text{Im}Z & -\text{Re}Z \end{array} \right] + \left[ \begin{array}{cccc} \epsilon_1 & \epsilon_2 & 0 & 0 \\ \epsilon_2 & \epsilon_3 & 0 & 0 \\ 0 & 0 & \epsilon_5 & \epsilon_4 \\ 0 & 0 & \epsilon_4 & -\epsilon_5 \end{array} \right],
\]

\[\text{Im}Z \neq 0, \quad S + \text{Re}Z = 0; \quad \epsilon_1 + \epsilon_2 + 2\epsilon_4 = 0.\]

Type \(I_R\): In this case \(C_0\) is given by (3.5) and one finds that the vector space tangent to the orbit of \(C_0\) at \(C_0\) consists of all vectors of the form

\[
\left( \begin{array}{cccc} 0 & -l_1(S_1 - S_2) & -l_2(S_1 - S_2) & l_2(S_1 - T) \\ -l_1(S_1 - S_2) & 0 & -l_4(S_2 - S_3) & l_3(S_2 - T) \\ -l_2(S_1 - S_2) & -l_4(S_2 - S_3) & 0 & l_6(S_3 - T) \\ l_1(S_1 - T) & l_4(S_2 - T) & l_6(S_3 - T) & 0 \end{array} \right).
\]
First we deal with type \([S, -S, -S, -T]\). From (3.19) we deduce that now \(\dim(\text{SO}^\prime C_0) = 6\), and the orbit tangent space at \(C_0\) is spanned by the vectors \(C_1, \ldots, C_5\) of (3.14) and

\[
C_b := \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}.
\]

Then the orthogonal complement of the orbit tangent space at \(C_0\) consists of all vectors of the form

\[
\varepsilon_1, \varepsilon_2, \varepsilon_3 \in \mathbb{R}, \quad \varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 = 0.
\]

Consequently, \(\varepsilon_1, \varepsilon_2, \varepsilon_3\) are the miniversal parameters \((m = 9 - 6 = 3)\) and finally:

\[
[S, -S, -S, -T]^{(e)}: \begin{pmatrix}
S_1 & 0 & 0 & 0 \\
0 & S_2 & 0 & 0 \\
0 & 0 & S_3 & 0 \\
0 & 0 & 0 & -T
\end{pmatrix}
\]

\[
\begin{pmatrix}
\varepsilon_1 & 0 & 0 & 0 \\
0 & \varepsilon_2 & 0 & 0 \\
0 & 0 & \varepsilon_3 & 0 \\
0 & 0 & 0 & -\varepsilon_4
\end{pmatrix},
\]

\(S_1, S_2, S_3, T\) distinct, \(S_1 + S_2 + S_3 + T = 0\), \(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 = 0\).

(3.22)

In the case of \([S - S_2 - 2T]_3\) (\(S_3 = T\)), the space tangent to the orbit of \(C_0\) at \(C_0\) is spanned by \(C_1, \ldots, C_5\), i.e., \(\dim(\text{SO}^\prime (3,1;\mathbb{R})\cdot C_0) = 5\) and \(m = 9 - 5 = 4\). The orthogonal complement consists of the following vectors:

\[
\begin{pmatrix}
\varepsilon_1 & 0 & 0 & 0 \\
0 & \varepsilon_2 & 0 & 0 \\
0 & 0 & \varepsilon_3 & \varepsilon_4 \\
0 & 0 & \varepsilon_4 & -\varepsilon_3
\end{pmatrix},
\]

\(\varepsilon_1, \ldots, \varepsilon_5 \in \mathbb{R}, \quad \varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 + \varepsilon_5 = 0\)

(3.23)

and the family \([S - S_2 - 2T]_3^{(e)}\) takes the form:

\[
[S - S_2 - 2T]_3^{(e)}: \begin{pmatrix}
S_1 & 0 & 0 & 0 \\
0 & S_2 & 0 & 0 \\
0 & 0 & T & 0 \\
0 & 0 & 0 & -T
\end{pmatrix}
\]

\[
\begin{pmatrix}
\varepsilon_1 & 0 & 0 & 0 \\
0 & \varepsilon_2 & 0 & 0 \\
0 & 0 & \varepsilon_3 & \varepsilon_4 \\
0 & 0 & \varepsilon_4 & -\varepsilon_3
\end{pmatrix},
\]

\(S \neq T, \quad S + T = 0; \quad \varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 + \varepsilon_5 = 0\).

(3.24)

For the type \([S - S_2 - T]_3\) the space tangent to the orbit of \(C_0\) at \(C_0\) is spanned by \(C_1, C_2, C_3, C_4, C_5, C_6\); \(\dim(\text{SO}^\prime (3,1;\mathbb{R})\cdot C_0) = 5\) and \(m = 9 - 5 = 4\). Then we find the family \([S - S_2 - T]_3^{(e)}\) to be:

\[
[S - S_2 - T]_3^{(e)}: \begin{pmatrix}
S_1 & 0 & 0 & 0 \\
0 & S_2 & 0 & 0 \\
0 & 0 & 0 & -T
\end{pmatrix}
\]

\[
\begin{pmatrix}
\varepsilon_1 & 0 & 0 & 0 \\
0 & \varepsilon_2 & \varepsilon_3 & 0 \\
0 & \varepsilon_3 & 0 & \varepsilon_4 \\
0 & 0 & -\varepsilon_4 & 0
\end{pmatrix},
\]

\(T \neq S_1 \neq S_2 \neq T, \quad S_1 + S_2 + T = 0, \quad \varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 = 0\).

(3.25)

For \([S - 3T]_2\) the space tangent to the orbit of \(C_0\) at \(C_0\) is spanned by \(C_1, C_2, C_3, C_4, C_5, C_6\). Hence, \(\dim(\text{SO}^\prime (3,1;\mathbb{R})\cdot C_0) = 3\) and \(m = 9 - 3 = 6\). The formula for the family \([S - 3T]_2^{(e)}\) is given by

\[
[S - 3T]_2^{(e)}: \begin{pmatrix}
S & 0 & 0 & 0 \\
0 & S & 0 & 0 \\
0 & 0 & T & 0 \\
0 & 0 & 0 & -T
\end{pmatrix}
\]

\[
\begin{pmatrix}
\varepsilon_1 & \varepsilon_2 & 0 & 0 \\
\varepsilon_2 & \varepsilon_3 & 0 & 0 \\
0 & 0 & \varepsilon_4 & \varepsilon_5 \\
0 & 0 & \varepsilon_5 & -\varepsilon_6
\end{pmatrix},
\]

\(S \neq T, \quad S + T = 0; \quad \varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 + \varepsilon_5 + \varepsilon_6 = 0\).

(3.26)

For \([3S - T]_2\) the space tangent to the orbit of \(C_0\) at \(C_0\) is spanned by \(C_1, C_2, C_3, C_4, C_5, C_6\). Thus \(\dim(\text{SO}^\prime (3,1;\mathbb{R})\cdot C_0) = 3\) and \(m = 9 - 3 = 6\). Then we get the following formula for the family \([3S - T]_2^{(e)}:\)

\[
[S - 3T]_2^{(e)}: \begin{pmatrix}
S & 0 & 0 & 0 \\
0 & T & 0 & 0 \\
0 & 0 & T & 0 \\
0 & 0 & 0 & -T
\end{pmatrix}
\]

\[
\begin{pmatrix}
\varepsilon_1 & 0 & 0 & 0 \\
0 & \varepsilon_2 & \varepsilon_3 & 0 \\
0 & \varepsilon_3 & \varepsilon_5 & \varepsilon_6 \\
0 & \varepsilon_4 & \varepsilon_6 & -\varepsilon_7
\end{pmatrix},
\]

\(S \neq T, \quad S + T = 0; \quad \varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 + \varepsilon_5 + \varepsilon_6 + \varepsilon_7 = 0\).

(3.27)
\( m = 9 - 0 = 9 \). The family \([4T]\) is given by

\[
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{pmatrix} +
\begin{pmatrix}
\epsilon_1 & \epsilon_2 & \epsilon_3 & \epsilon_4 \\
\epsilon_2 & \epsilon_3 & \epsilon_4 & 0 \\
\epsilon_3 & \epsilon_4 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{pmatrix}.
\]

\[
\begin{pmatrix}
\epsilon_1 + \epsilon_3 + \epsilon_6 + \epsilon_7 = 0.
\end{pmatrix}
\]

Type II: In this case \( C_0 \) is given by (3.6), and with the use of (3.9) and (3.10) one finds that the orbit tangent space at \( C_0 \) consists of the following vectors:

\[
\begin{pmatrix}
l_1(S_1 - S_2) - l_3(S_1 - N - \rho) + l_4\rho \\
l_1(S_1 - S_2) - l_4(S_2 - N - \rho) + l_3\rho \\
l_1(S_1 - S_2 - N + \rho) + l_3\rho & l_4(S_2 - S_2 - N + \rho) + l_4\rho \\
\end{pmatrix}.
\]

Consider now the type \([S - 3N]\). The vector space tangent to the orbit of \( C_0 \) at \( C_0 \) is spanned by [see (3.30) for \( S_2 = N \neq S_1 \)] \( C_1, C_2, C_3, C_4, C_5, C_6 + C_9 \); \( \dim(\mathfrak{so}(3,1;\mathbb{R})\cdot C_0) = 6 \), \( m = 9 - 6 = 3 \). Then the family \([S_1 - S_2 - 2N]\) is defined by

\[
\begin{pmatrix}
S_1 & 0 & 0 & 0 \\
0 & S_2 & 0 & 0 \\
0 & 0 & N + \rho & \rho \\
0 & 0 & \rho & - N + \rho \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
\epsilon_1 & 0 & 0 & 0 \\
0 & \epsilon_2 & 0 & 0 \\
0 & 0 & \epsilon_3 & \frac{1}{2}(\epsilon_4 - \epsilon_3) \\
0 & 0 & \frac{1}{2}(\epsilon_4 - \epsilon_3) & - \epsilon_4 \\
\end{pmatrix},
\]

\[
\rho \neq 0, \ N \neq S_1 \neq S_2 \neq N, \ S_1 + S_2 + 2N = 0, \\
\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4 = 0.
\]

In the case of the type \([2S - 2N]\), the orbit tangent space at \( C_0 \) is spanned by \( C_1, C_2, C_3, C_4, C_5, C_6 + C_9 \); \( \dim(\mathfrak{so}(3,1;\mathbb{R})\cdot C_0) = 5 \) and \( m = 9 - 5 = 4 \). Thus for the family \([2S - 2N]\) one gets

\[
\begin{pmatrix}
S & 0 & 0 & 0 \\
0 & S & 0 & 0 \\
0 & 0 & N + \rho & \rho \\
0 & 0 & \rho & - N + \rho \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
\epsilon_1 & \epsilon_2 & 0 & 0 \\
\epsilon_2 & \epsilon_3 & 0 & 0 \\
0 & 0 & \epsilon_4 & \frac{1}{2}(\epsilon_5 - \epsilon_4) \\
0 & 0 & \frac{1}{2}(\epsilon_5 - \epsilon_4) & - \epsilon_5 \\
\end{pmatrix},
\]

\[
\rho \neq 0, \ S \neq N, \ S + N = 0, \\
\epsilon_1 + \epsilon_3 + \epsilon_4 + \epsilon_5 = 0.
\]

Type III: Here \( C_0 \) is given by (3.7). Then by (3.9) and (3.10) one finds that the space tangent to the orbit of \( C_0 \) at \( C_0 \) consists of the following vectors:
In the case of \([S - 3N]_4\), from (3.35) one deduces that the orbit tangent space at \(C_0\) is spanned by the vectors \(C_1, C_2, C_3, C_4 + C_5, C_6 + C_0,\) and \(C_6 + 2C_4,\) where

\[
C_0 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.36}
\]

Hence, \(\dim(SO'(3,1;\mathbb{R}) \cdot C_0) = 6\) and \(m = 9 - 6 = 3.\) Finally, for \([S - 3N]_4^{(e)}\) we get

\[
[S - 3N]^{(e)}_4 = \begin{pmatrix} S & 0 & 0 & 0 \\ 0 & N & \rho & \rho \\ 0 & \rho & N & 0 \\ 0 & 0 & 0 & -N \end{pmatrix}
\]

\[
\begin{pmatrix} \epsilon_1 & 0 & 0 & 0 \\ -\frac{1}{\epsilon_1} \epsilon_1 & \epsilon_2 & -\epsilon_2 \\ 0 & \epsilon_2 & \epsilon_3 & \frac{1}{2}(\epsilon_4 - \epsilon_3) \\ 0 & -\epsilon_2 & \frac{1}{2}(\epsilon_4 - \epsilon_3) & -\epsilon_4 \end{pmatrix},
\]

\[
\rho \neq 0, \ S \neq N \ S \neq 3N = 0, \ 3\epsilon_1 = \epsilon_3 = \epsilon_4 = 0. \tag{3.37}
\]

Consider now the type \([4N]_3.\) The orbit tangent space at \(C_0\) is spanned by \(C_1, C_2 + C_3, C_4 + C_5, C_6 + C_0,\) and \(C_6 + 2C_4; \dim(SO'(3,1;\mathbb{R}) \cdot C_0) = 5,\) and \(m = 9 - 5 = 4.\)

The family \([4N]^{(e)}_3\) is given by

\[
[4N]^{(e)}_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \rho & \rho \\ 0 & \rho & 0 \\ 0 & 0 & 0 \end{pmatrix}
\]

\[
\begin{pmatrix} \epsilon_1 & 0 & -\epsilon_2 \\ -\frac{1}{\epsilon_1} \epsilon_1 & \epsilon_3 & -\epsilon_3 \\ 0 & \epsilon_3 & \epsilon_4 & \frac{1}{2}(\epsilon_3 - \epsilon_4) \\ 0 & -\epsilon_3 & \frac{1}{2}(\epsilon_3 - \epsilon_4) & -\epsilon_4 \end{pmatrix},
\]

\[
\rho \neq 0, \ S \neq N \ S \neq 3N = 0, \ 3\epsilon_1 = \epsilon_3 = \epsilon_4 = 0. \tag{3.38}
\]

The miniversal deformations we have just found enable us to show instability of the algebraic classification of the traceless Ricci tensor \(\mathcal{C}\) under small perturbations. Let \(C_0\) be a canonical matrix of some algebraic type which we denote by \([\ldots]\), and let \(R^m \ni \epsilon \rightarrow C(\epsilon) \in [\ldots]^{(e)}\) be a miniversal deformation of \(C_0.\) Then the type \([\ldots]_{\ldots}\) is said to be deformable by a small perturbation to a type, say \([\ldots]\) if for every real \(r > 0\) there exists \(\epsilon' = (\epsilon'_1, \ldots, \epsilon'_m) \in \mathbb{R}^m\) such that \((\epsilon'_1^2 + \cdots + \epsilon'_m^2)^{1/2} < r\) and the matrix \(C(\epsilon')\) is of the type \([\ldots].\) This deformability will be denoted by the symbol \(\triangleright,\) i.e., we write \(\ldots \triangleleft [\ldots].\) Careful analysis of formulas describing the miniversal deformations of canonical matrices for the algebraic types of \(\mathcal{C}\) leads to the following scheme for all possible deformations under small perturbations:

\[
[S - 3T]_{4} \triangleright [S - 3T], [S - 3N], [S - 2S - 2T], [S - S - 3 - T],
\]

\[
[S - 3T]_{3} \triangleright [S - 3T], [S - 2S - 2T], [S - 2S - 2T], [S - 2S - 2T - T], [S - S - 2S - 2T], [S - S - 2S - 2T], [S - S - 2S - 2T]
\]

\[
[S - 3N]_{4} \triangleright [S - 3N], [S - 2S - 2T], [S - S - 2S - 2T], [S - S - 2S - 2T]
\]

\[
[S - 2S - 2T]_{3} \triangleright [S - 2S - 2T], [S - 2S - 2T], [S - 2S - 2T], [S - 2S - 2T - T], [S - S - 2S - 2T], [S - S - 2S - 2T], [S - S - 2S - 2T],
\]

\[
[S - S - 2S - 2T]_{3} \triangleright [S - S - 2S - 2T], [S - S - 2S - 2T], [S - S - 2S - 2T], [S - S - 2S - 2T], [S - S - 2S - 2T], [S - S - 2S - 2T], [S - S - 2S - 2T], [S - S - 2S - 2T], [S - S - 2S - 2T],
\]

\[
[S - S - 2S - 2T]_{2} \triangleright [S - S - 2S - 2T], [S - S - 2S - 2T], [S - S - 2S - 2T], [S - S - 2S - 2T], [S - S - 2S - 2T], [S - S - 2S - 2T], [S - S - 2S - 2T], [S - S - 2S - 2T], [S - S - 2S - 2T], [S - S - 2S - 2T],
\]

\[
[S - S - 2S - 2T]_{1} \triangleright [S - S - 2S - 2T], [S - S - 2S - 2T], [S - S - 2S - 2T], [S - S - 2S - 2T], [S - S - 2S - 2T], [S - S - 2S - 2T], [S - S - 2S - 2T], [S - S - 2S - 2T], [S - S - 2S - 2T], [S - S - 2S - 2T],
\]

\[
[4T]_{1} \triangleright \text{any type};
\]
Now one can interpret our results as follows: (i) Formulas defining the miniversal deformations give the stable canonical forms of $\mathcal{C}$ if the miniversal parameters $\epsilon_1, \ldots, \epsilon_m$ are assumed to be sufficiently small, (ii) The algebraic types of the stable canonical forms are presented by the scheme (3.39).

The scheme (3.39) enables us to find the degeneration scheme for algebraic types of $\mathcal{C}$ as the degeneration is an inverse relation to the deformation. This degeneration scheme is given by Fig. 1, where the degeneration is represented by the arrow $\rightarrow$. The scheme (3.39) can be obtained from Fig. 1 by following the arrows backwards. (Compare Fig. 1 with Refs. 4, 5, 7, 8).

One finds that the types $[S, -S, -2N]_4$, $[S, -S, -S, -T]_4$, and $[S, -S, -Z - \overline{Z}]_4$ are stable under small perturbations. (Evidently, only these types are stable).

IV. FINAL REMARKS

Einstein equations

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} - \Lambda g_{\mu\nu} = -8\pi T_{\mu\nu}$$

(4.1)
yield

$$T_{\mu\nu} = - \left( \frac{1}{8\pi} \right) C_{\mu\nu} + \frac{4}{\tau} g_{\mu\nu},$$

(4.2)

where

$$\tau = T^\nu_\nu = \left( \frac{1}{8\pi} \right) \cdot \left( R + 4\Lambda \right).$$

(4.3)

From (4.2) it follows that the algebraic classification of $T_{\mu\nu}$ is defined by the algebraic classification of $C_{\mu\nu}$. Then, formulas (3.5) and (4.2) or (3.6) and (4.2) enable us to express the energy conditions (see the Introduction) for Types $I_R$ or $II$, respectively, in terms of eigenvalues of $T_{\mu\nu}$. Straightforward calculations lead to the following results (compare Refs. 4, 9, 13, 15):

(1) The weak energy condition (1.1a):

Type $I_R$: $t < 0$ and $t \leq s_1$, $r = 1, 2, 3,$

(4.4)

Type $II$: $\rho < 0$, $n \leq s_1$, $l = 1, 2.$

(4.5)

(2) The dominant energy condition (1.1a), (1.1b):

Type $I_R$: $t < s_1$, $r = 1, 2, 3,$

(4.6)

Type $II$: $\rho < 0$, $n \leq s_1$, $l = 1, 2.$

(4.9)

(3) The null convergence condition (1.2);

Type $I_R$: $t < s_1$, $r = 1, 2, 3,$

(4.8)

Type $II$: $\rho < 0$, $n \leq s_1$, $l = 1, 2.$

(4.9)

(4) The timelike convergence condition (1.3);

Type $I_R$: $t < s_1$, and $t \leq s_1 + s_2 + s_3 - (1/4\pi)\Lambda$, $r = 1, 2, 3,$

(4.10)

Type $II$: $\rho < 0$, $n \leq s_1$, and $0 \leq s_1 + s_2 + s_3 - (1/4\pi)\Lambda$, $l = 1, 2,$

(4.11)

are the eigenvalues of $T_{\mu\nu}$ for Type $I_R$ and Type $II$, respectively.

It is well known that none of the energy conditions (1)–(4) can be satisfied by $T_{\mu\nu}$ defined according to (4.1)–(4.3) if the traceless Ricci tensor $C_{\mu\nu}$ is of Type $I_R$ or $III.$ But, as the results of Sec. III show [see (3.39) and Fig. 1] each type except $[3S - T]_2$, $[S_1 - T]_2$, and $[S_1 - S_2 - T]_4$, appears to be deformable by small perturbations to “unphysical” Types $I_R$ and $III.$ Consequently, for all types of $C_{\mu\nu}$ except $[3S - T]_2$, $[S_1 - T]_2$, and $[S_1 - S_2 - T]_4$, the energy conditions (1)–(4) are unstable under small perturbations. What concerns the exceptional types, one can show easily that here the energy conditions are stable if and only if strict inequalities in (1)–(4) [or (4.4), (4.6), (4.8), (4.10), resp.] are satisfied. Thus, for example, in the case of perfect fluid $T_{\mu\nu} = (\epsilon + p) u_\mu u_\nu + pg_{\mu\nu}$, $C_{\mu\nu}$ appears to be of the type $[3S - T]_2$, and the stable energy conditions read:

(1) $\epsilon > 0$, $\epsilon < \rho$, $2$, $-\epsilon < \rho < \epsilon$, $3$, $-\epsilon < \rho$.

(4) $\epsilon < \rho$, $\epsilon > 3p - 1/4\pi\Lambda$, respectively.

Consequences of the results obtained in this paper in the theory of singularities we intend to consider elsewhere.

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