Conformal extensions of the Galilei group and their relation to the Schrödinger group

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Various authors have considered a conformal extension $C_G$ of the Galilei group which in some sense is the nonrelativistic limit of the conformal extension of the Poincaré group, and have also established an invariance group for the free-particle Schrödinger equation, the "Schrödinger group." Here we establish the most general conformal extension $C_G$ of the Galilei group, which is found to be identical to the group of the most general coordinate transformations that permit the use of noninertial frames of reference and of curvilinear coordinates in Galilei-invariant theories, which was considered by one of us some time ago, and is a gauge group containing a number of arbitrary functions. Both $C_G$ and the Schrödinger group are subgroups of $C_G$ containing the Galilei group, but otherwise they do not overlap. The Hamilton–Jacobi and Schrödinger equations for particles which are free or interact via inverse-square potentials are shown to be invariant under the Schrödinger group, and a further invariance of the Hamilton–Jacobi equation is established.

I. INTRODUCTION

In 1909, Cunningham and Bateman realized that Maxwell's equations are invariant not only under the 10-parameter Poincaré group (= inhomogeneous Lorentz) group, but under the wider 15-parameter conformal group $C_P$. Since then, conformal invariance has been considered in many areas of physics, and in recent years has found renewed interest in high energy physics.

For our present purposes, the general conformal group $C$ is most concisely defined as the group of all transformations which in any Lorentz space with metric tensor $g_{\alpha\beta}$ locally leave the light cone invariant. However, in the following we shall mainly be interested in Minkowski space and its metric tensor $\eta_{\alpha\beta}$. The corresponding conformal group $C_P$ is more appropriately called the conformal extension of the Poincaré group; it is briefly discussed in Sec. III.

In connection with the renewed interest in conformal invariance in particle physics, a conformal extension of the Galilei group was considered in a study of Galilei-invariant field theories by Hagen and this group was studied in detail by Roman et al. Simultaneously, it was realized by Niederer that the Schrödinger equation for a free particle is invariant under a wider group of transformations (the "Schrödinger group") than the Galilei group, identical with the group considered by Hagen. The relation of this group to the conformal group was studied by Barut and Niederer, both of whom compared the Schrödinger group to the nonrelativistic limit $C_G$ of the conformal extension of the Poincaré group.

A similar study was undertaken for the Hamilton–Jacobi equation by Boyer and Peñafiel.

Our own interest in Galilean analogs to the conformal group $C_P$ arose from a continuing investigation of possible dynamics of interacting particles. In Sec. III we show that if such Galilean analogs are based on the nonrelativistic analog of Eq. (1), a group $C_G$ very much wider than that considered in Refs. 5–10 results, which is identical with a group considered by one of us some time ago in a different context, and is a gauge group containing a number of arbitrary functions. Even if we restrict it further than required by this analogy, we obtain a gauge group which is wider than the Schrödinger group. To obtain these results, it is convenient to use a formalism for the Galilei group introduced earlier, which is outlined in Sec. II. Both the latter and $C_G$ are subgroups of $C_G$ containing the Galilei group, but otherwise they do not overlap. In Sec. IV, we present a simple proof of the invariance of the Hamilton–Jacobi and Schrödinger equations for free particles or particles interacting via inverse-square potentials under the Schrödinger group as well as a further invariance of the Hamilton–Jacobi equation. The relation of our results to previous work is discussed in Sec. V.

II. UNIFIED TREATMENT OF THE POINCARE AND GALILEI GROUPS

We consider the linear group of transformations of the Cartesian space coordinates $x^1, x^2, x^3$, and the time $t = x^0$

$$x'^\mu = \alpha^\mu_{\nu} x^\nu + \xi^\mu,$$  \hspace{1cm} (1)

where the $\alpha^\mu_{\nu}$'s and $\xi^\mu$'s are constant parameters. Here and in the following, summation over repeated indices is understood, Greek indices always range from 0 to 3, and Roman ones from 1 to 3.

The Poincaré group is the group of transformation (1), restricted by the condition

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Equations (2P) and (3P) imply
\[ \eta^{\mu \nu} \varepsilon_{\nu} = \eta_{\mu}^\nu. \] (4P)

We shall take the nonvanishing components of these tensors to be
\[ \eta_{11} = \eta_{22} = \eta_{33} = -c^2, \]
\[ \eta^{00} = 1, \quad \eta^{01} = \eta^{02} = \eta^{03} = -c^2. \] (5P)

The full inhomogeneous Galilei group is the group of transformations (1), restricted by the conditions (3). The tensors \( \tilde{\varepsilon}_{\mu}^\nu \) and \( \tilde{\varepsilon}^{\mu \nu} \) are singular; we can choose as their nonvanishing components
\[ \tilde{\varepsilon}_{00}^\nu = 1, \]
\[ \tilde{\varepsilon}^{00} = 1, \quad \tilde{\varepsilon}^{11} = \tilde{\varepsilon}^{22} = \tilde{\varepsilon}^{33} = -1, \] (6P)
and thus
\[ \tilde{\varepsilon}_{00}^\nu \varepsilon_{\nu} = 0, \] (3G)

Clearly \( \tilde{\varepsilon}_{\mu}^\nu \) and \( \tilde{\varepsilon}^{\mu \nu} \) are the limits \( c \rightarrow \infty \) of \( \eta_{\mu}^\nu \) and \( c^{-2} \eta^{\mu \nu} \), respectively; since they are independent, so are the relations (2G) and (4G).4

Equations (2), (4)–(6) imply that the Jacobian \( J \) of transformation (1) equals \( 1 \) in both cases. Thus both the Poincaré and the Galilei group consist of four parts, corresponding to the four combinations of the signs of \( J \) and of \( \alpha^\mu_0 \). The part with \( J = \text{sgn} \alpha^0_0 = 1 \) forms a subgroup, the proper orthochronous Poincaré and Galilei group, respectively.

The space of the Poincaré group is metric, with a metric tensor \( \eta_{\mu \nu} \), and a four-dimensional infinitesimal distance defined by
\[ ds^2 = \eta_{\mu \nu} dx^\mu dx^\nu. \] (7P)

For the Galilei group, we could also introduce such a distance through
\[ ds^2 = \tilde{\varepsilon}_{\mu}^\nu \varepsilon_{\nu} dx^\mu dx^\nu. \] (7G)

However, the “metric” \( \tilde{\varepsilon}_{\mu}^\nu \) is singular, and thus the space is not Riemannian; the separation (7G) is a pure time interval, and assigns a separation zero to any two simultaneous events.

Unlike \( \eta_{\mu \nu} \) and its inverse \( \eta^{\mu \nu} \), \( \tilde{\varepsilon}_{\mu}^\nu \) and \( \tilde{\varepsilon}^{\mu \nu} \) cannot be used to lower and raise indices reversibly, and in general co- and contravariant vectors are distinct quantities. Since the Christoffel symbols and the curvature tensor defined from \( \eta_{\mu \nu} \) vanish, the metric space characterized by \( \eta_{\mu \nu} \) is flat. No analogous statements can be made for the space characterized by \( \tilde{\varepsilon}_{\mu}^\nu \) however, if we introduce vanishing affine connections \( \Gamma_{\mu}^\nu_\rho \) by definition, the corresponding curvature tensor also vanishes, and thus this affinely connected space also is flat.

\section*{III. CONFORMAL EXTENSIONS OF THE POINCARÉ AND GALILEI GROUPS}

The conformal extension \( C_p \) of the Poincaré group is the group of all coordinate transformation
\[ x^\mu = x^\mu (x') \] (8)
that connect line elements of the form
\[ ds^2 = \phi^{-2}(x') \eta_{\mu \nu} dx^\mu dx^\nu = \eta_{\mu \nu} dx^\mu dx^\nu \] (9P)
with each other,13 and thus preserve the light cones \( ds^2 = 0 \). Clearly, the Poincaré transformations form a subgroup; another subgroup is that of the scale transformations (dilatations)
\[ x^\mu = C^{-1} x^\mu. \] (10P)

It can be shown that the most general conformal transformation is the product of a Poincaré transformation and a “Haantjes transformation” (product of dilatations and acceleration transformations)
\[ x^\mu = \frac{x^\mu - C^{-1} \eta_{\rho \sigma} x^\rho x^\sigma}{C - 2 \eta^{\rho \sigma} x^\rho x^\sigma + C \eta_{\rho \sigma} \phi^{-2} \eta^{\rho \sigma} x^2}, \] (11P)
where \( C \) and \( \phi \) are five arbitrary constant parameters, and thus the conformal extension of the Poincaré group is a 15-parameter group. The Galilean limit of this transformation (the “Galilean Haantjes transformation”) is
\[ x^\mu = \frac{x^\mu - C^{-1} \eta_{\rho \sigma} x^\rho x^\sigma}{C - 2 \eta^{\rho \sigma} x^\rho x^\sigma + C \eta_{\rho \sigma} \phi^{-2} \eta^{\rho \sigma} x^2}, \] (11GA)
which from Eq. (5G) is equivalent to
\[ t' = \frac{t}{C - t^2}, \quad x' = \frac{C x - x^2}{(C - t^2)^2}. \] (11Gb)

This set of transformations together with the Galilei transformations forms a 15-parameter group \( C_G \), which has a structure very similar to that of the conformal extension of the Poincaré group, and has therefore been considered occasionally as the appropriate definition of the Galilean conformal group.8 It is, however, by no means the most general conformal extension of the Galilei group.

Before proceeding with a study of this extension, we note that \( C_p \) in the interpretation adopted here is to be understood as a group of transformations on the coordinates, but not on the metric tensor. Therefore, \( ds^2 \) is not an invariant and \( \tilde{\eta}_{\mu \nu} \) does not equal \( \eta'_{\mu \nu} \), but instead is given by
\[ \tilde{\eta}_{\mu \nu} = \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial x'^\sigma}{\partial x^\nu} \eta_{\rho \sigma} - \phi^{-2} \eta_{\mu \nu}, \] (12P)
where the factor of \( \eta_{\mu \nu} \) arises from the transformation of \( dx^\mu dx^\nu \), i.e., the expression (9P) arises from \( \eta_{\mu \nu} dx^\mu dx^\nu \) rather than \( \eta'_{\mu \nu} dx'^\mu dx'^\nu \). A similar interpretation must be adopted for the transformations of the conformal extension of the Galilei group.

We can define a “contravariant” \( \tilde{\eta}'_{\mu \nu} \) as the inverse of \( \tilde{\eta}_{\mu \nu} \) from a relation corresponding to (3P) to obtain
\[ \tilde{\eta}'_{\mu \nu} = \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial x'^\sigma}{\partial x^\nu} \eta'_{\rho \sigma} = \phi^2 \eta_{\mu \nu}, \] (13P)
which does not equal \( \eta'_{\mu \nu} \).
To obtain the most general extension of the Galilei group, we proceed directly from the Galilean analog of preservation of light cones. In the limit $c\to\infty$ the light cones $ds^2=0$ with $ds^2$ given by (7P) degenerate to planes of constant time (absolute simultaneity), for which $ds^2=0$ with the Galilean $ds^2$ (7G). The most general transformations $C_G$ maintaining this condition are

$$x'^\alpha = x'^\alpha(x^\beta), \quad dx'^\alpha/dx^\beta > 0 \quad \text{for all } x^\alpha,$$

or

$$< 0 \quad \text{for all } x^\alpha,$$

(14Ga)

$$x'^m = x'^m(x^\epsilon).$$

(14Gb)

Clearly, these transformations contain both the Galilei group and the group (11G) as special cases, but are much more general.

With the interpretation adopted above, we now have

$$ds^2 = \phi^{-2}(x^\alpha)\delta_{\alpha\beta}dx^\alpha dx^\beta = \phi^{-2}\delta_{\alpha\beta}dx^\alpha dx^\beta,$$

(9G)

$$\delta_{\alpha\beta} \overset{\partial x^\alpha}{\partial x'^\alpha} \overset{\partial x^\beta}{\partial x'^\beta} = \phi^{-2}\delta_{\alpha\beta}.$$

(12G)

However, if we wish to define a "contravariant" $\delta^\alpha_\beta$ from a relation corresponding to (3G) in analogy to the procedure used above to obtain (13P), we only get

$$\delta^\alpha_\beta = \omega^2(x^\lambda)\overset{\partial x^\alpha}{\partial x^\beta} \overset{\partial x^\epsilon}{\partial x^\epsilon}.$$

(13G)

which is not necessarily proportional to $\delta^\alpha_\beta$ and contains an arbitrary factor $\omega^2(x^\lambda)$ because of the degenerate form of (3G). However, because of the form (14G) of the coordinate transformations we have at least

$$\delta^\alpha_\beta = \omega^2 = 0.$$

(15G)

The relations (14G) are precisely those obtained in Ref. 12 as the most general coordinate transformations allowed that permit the use of noninertial frames of reference and curvilinear coordinates without changing the physical content of Galilei-invariant theories. The only restriction on (analytical) coordinate transformations imposed there was the exclusion of coordinate systems for which signals emitted at a time $t_0$ could arrive at some points of the systems at $t > t_0$ and at others at $t < t_0$.

It should be noted that imposition of the corresponding restriction on coordinate transformations for Poincaré-invariant theories does not lead to the conformal extension of the Poincaré group $C_P$. The condition on the description of signals stated above implies (in addition to preservation of light cones) that the space- or timelike character of separations [i.e., the sign of $ds^2$ in (9P)] is maintained, a condition not satisfied by the acceleration transformations. This condition leads to a set of restrictions on the transformed metric tensor $\delta^\alpha_\beta$ [4,11,12]. In the Galilei case, no such additional condition is implied, due to the collapse of the cone to a plane.

Some time ago, Zeeman [26] showed that the requirement of preservation of light cones in Minkowski space and of orientation of timelike vectors implies the "causality group," defined as the product of the orthochronous Poincaré group and the dilatation group. This restriction actually is an immediate consequence of the long-known fact that $C_P$ is the widest group of transformations in Minkowski space which preserves the light cones, but that the subgroups of acceleration transformations and of antichronous Poincaré transformations do not preserve time orientation. A requirement of "causality" for Newtonian space–time analogous to Zeeman's for Minkowski space would demand preservation of absolute simultaneity and of time orientation, and thus the subgroup of orthochronous transformations of $C_G$ defined by (14).

Thus Zeeman's statement "causality implies the Lorentz group" is valid only in Minkowski space; furthermore, as already discussed in Ref. 12 (Footnote 49) in connection with the transformations (14Ga), it is too strong a requirement to demand preservation of time orientation, "since this would assign physical meaning to the obviously conventional orientation of the time axis..." Allowing both signs does not contradict the "causality condition" that a signal should not arrive earlier than it was emitted, which can be looked upon as a definition either of "signal" or of "earlier."

Therefore antichronous transformations need not be excluded, and the physically required causality conditions do not impose any restrictions on $C_G$, and in the case of $C_P$ only exclude the acceleration transformations and impose the restrictions on $\delta^\alpha_\beta$, mentioned above.

Because of the difference between the relation (13P) and (13G) there is a clear qualitative difference between the group of transformations $C_G$ allowed by a conformal extension of the Galilei group and the group $C_{G_P}$ obtained as the Galilean limit of $C_P$. On the other hand, we can subject the transformations of $C_G$ to arbitrary restrictions to achieve a closer similarity to, or even identity with, the group $C_{G_P}$.

The weakest restriction on the transformations (14G) that reduces Eq. (13G) to a form reminiscent of (13P) is the requirement

$$\delta^\alpha_\beta = \Omega^2(x^\lambda)\delta^\alpha_\beta.$$

(16G)

This only restricts the transformations (14Gb), but not (14Ga). From (13G) and (14G) we obtain

$$\frac{\partial x'^m}{\partial x^m} \frac{\partial x'^n}{\partial x^n} = \frac{\Omega^2}{\omega^2} \delta^m_n,$$

(17a)

which implies

$$\frac{\partial x'^m}{\partial x^m} = \frac{\Omega^2}{\omega^2} \delta^m_n;$$

(17b)

no restrictions are imposed on $\partial x'^m/\partial x^m$.

The most general transformation satisfying the condition (17Gb) is

$$x'^m = F(x^\lambda)\alpha^m_\alpha(x^\beta)x^\beta + \alpha^m_\alpha(x^\beta)(\epsilon^\beta\gamma(x^\lambda)x^\gamma + \xi^m(x^\lambda)), \quad \text{(18G)}$$

where

$$\alpha^m_\alpha\delta^\alpha_\beta = \delta^m_n,$$

(18b)

which together with Eq. (14Ga) defines a group $C_{G_P}$, and for which


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\begin{equation}
\frac{\Omega^2(\xi')}{\omega^2(\xi')} = \frac{F^2(x')}{(1 + 2C x' + \xi^2)^2}.
\end{equation}

(19G)

The following distinct subgroups of the transformations (14Ga) and (18G), corresponding to simple forms of the functions $F$ and $\xi^m$, are easily recognized [where we always first state the transformation (14Ga), written in terms of the time variables, and then the values of some of the functions and parameters appearing in (18Ga); those not specified explicitly are unrestricted constants, and all quantities not given explicitly as functions of $t$ are understood to be constants]:

I. The Galilei group:
\begin{align*}
t' &= t + \xi^2; \\
F &= 1, \quad \xi^m = 0, \quad \xi^n = \alpha^n_\phi + \xi^n_\phi.
\end{align*}

II. The Galilean Haantjes transformation (11G):
\begin{align*}
t' &= \frac{t}{C - \xi^2}; \\
F &= \frac{C}{(C - \xi^2)^2}, \quad \alpha^n_\phi = \delta^n_\phi, \\
\xi' &= 0, \quad \xi^n = -\frac{\xi^2}{(C - \xi^2)}.\end{align*}

III. The three-dimensional conformal transformation:
\begin{align*}
t' &= t; \\
F &= 1, \quad \alpha^n_\phi = \xi^n_\phi, \quad \xi^n = 0.
\end{align*}

IV. The "Schrödinger dilatation"1:
\begin{align*}
t' &= C \xi^2; \\
F &= C^2, \quad \alpha^n_\phi = \delta^n_\phi, \quad \xi' = 0, \quad \xi^n = 0.
\end{align*}

V. The "Schrödinger expansion"1:
\begin{align*}
t' &= Ft; \\
F &= (1 - t^2)^{-1}, \quad \alpha^n_\phi = \delta^n_\phi, \quad \xi' = 0, \quad \xi^n = 0.
\end{align*}

Clearly there are many more subgroups. In particular, it should be noted that since any dilatations of $x^4$ and of the $x^n$ are independent, their ratio is arbitrary, and thus the dilatation subgroup of the Haantjes transformations and the Schrödinger dilatations are only two particular cases of another subgroup of $C_\phi$ (overlapping the subgroup II and containing IV):

VI. The general dilatations:
\begin{align*}
t' &= B^{-1}t; \\
F &= D^{-1}, \quad \alpha^n_\phi = \xi^n_\phi, \quad \xi' = 0, \quad \xi^n = 0.
\end{align*}

We can further restrict our transformations by requiring in Eq. (16G)
\begin{equation}
\Omega^2 = \phi(\xi^2).
\end{equation}

(20Ga)

However, this still leaves an arbitrariness beyond that of $C_\phi$ because of the presence of the arbitrary function $\omega(x^4)$, and indeed imposes no restriction whatever on the transformation (18G). To obtain the full Galilean analogue to Eq. (13P), we must require in addition to (20Ga) that
\begin{equation}
\omega(x^4) = 1,
\end{equation}

(20Gb)

since these relations imply
\begin{equation}
\frac{\partial \phi}{\partial t} + \frac{\partial \psi}{\partial x^4} x^4 \phi = 0.
\end{equation}

(21G)

the difference between (13P) and (21G) then (apart from the fact that $\psi_{xx}$ is degenerate) is that in the Galilean case $\phi$ can only be a function of $x^4$ alone.

Therefore Eqs. (20G) and (19G) require that $\xi'$ vanishes, and thus the transformations (18G) reduces to
\begin{equation}
x'^m = F(x^0, \alpha^n_\phi, x^n) + \xi'^m(x^4),
\end{equation}

(22G)

This is the group of orthogonal coordinate systems undergoing arbitrary accelerations as well as time-dependent dilatations. It, together with the time transformations (14Ga), forms a group $C_\phi$, the "Léibniz group" recently discussed by Barbour and Bertotti in a different context.11 $C_\phi$ includes the subgroups I, IV, and V listed above, but both the Galilean acceleration transformation and the three-dimensional conformal transformation are excluded. The product of these three subgroups is the Schrödinger group $C_s$, the Galilei group defined by (14G) contains four bars of the functions and parameters appearing in (18Ga); those not specified explicitly are unrestricted constants, and all quantities not given explicitly as functions of $t$ are understood to be constants:}

\begin{align*}
t' &= t + \xi^2; \\
F &= C - \xi^2; \\
\xi' &= 0, \quad \xi^n = -\frac{\xi^2}{(C - \xi^2)}.\end{align*}

(23G)

[where all parameters are constants, and the $\alpha^n_\phi$'s are subject to conditions (18Gb), which thus is a subgroup both of the conformal extension $C_\phi$ of the Galilei group and of its subgroup $C_s$ restricted by Eq. (20G). However, it is not a subgroup of the group $C_\phi$ discussed above (the product of the Galilei transformation and the Galilean Haantjes transformations). To obtain this group, we can not require condition (20G), but must instead only demand (16G) and restrict the transformation group (18G) to the product of the subgroups I and II listed above.

As noted before, both $C_s$ and $C_\phi$ are 15-parameter groups. Since from Eq. (18Gb) only three of the $\alpha^n_\phi$ are independent, the Schrödinger group $C_s$ is a 12-parameter group. On the other hand, the conformal extension $C_\phi$ of the Galilei group defined by (14G) contains four arbitrary functions $x^4(x^0)$ and $x'^m(x^n)$ and its restricted forms defined by (14Ga) and (18G) or (20G) contain 11 or 8 arbitrary functions of $x^4$ alone, respectively, $[x^0, F, \xi^m, \alpha^n_\phi]$ in both cases, plus $\xi'$ in the case (18G); thus they all are gauge groups.

IV. THE INVARIANCE GROUPS OF THE SCHRODINGER AND THE HAMILTON-JACOBI EQUATION

As noted in the Introduction, a number of authors have recently investigated the invariance groups of the free-particle Schrödinger equation
\begin{equation}
-\frac{i}{\hbar} \frac{\partial \psi}{\partial t} + \frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} = 0
\end{equation}

(24)

and Hamilton-Jacobi equation
\begin{equation}
\frac{\partial S}{\partial t} + \frac{1}{2m} \frac{\partial S}{\partial x^2} + \frac{1}{2m} \frac{\partial S}{\partial x^2} = 0.
\end{equation}

(25)

All of these investigations of the Schrödinger equation
worked with Eq. (24) and with finite transformations. However, it is much more convenient to work with the variational principle \( \delta I = 0 \) for Eq. (24), where

\[
I = \int \left[ \frac{k}{2t} \left( \frac{\partial \phi^*}{\partial t} - \frac{\partial \phi}{\partial t} \right) - \frac{k^2}{2m} \frac{\partial \phi^*}{\partial x^i} \frac{\partial \phi}{\partial x^j} \right] d^4x,
\]

and with infinitesimal transformations.

Using the Galilean tensors discussed in Sec. II, Eqs. (25) and (26) can be written

\[
\frac{\partial S}{\partial t} - \frac{1}{2m} \mu^\nu \frac{\partial S}{\partial x^\nu} \frac{\partial S}{\partial x^\mu} = 0
\]

and

\[
I = \int \left[ \frac{k}{2t} \left( \frac{\partial \phi^*}{\partial t} - \frac{\partial \phi}{\partial t} \right) + \frac{k^2}{2m} \mu^\nu \frac{\partial \phi^*}{\partial x^\nu} \frac{\partial \phi}{\partial x^\mu} \right] d^4x.
\]

Before considering the various conformal extensions of the Galilei group introduced in Sec. III, it will be instructive to consider arbitrary coordinate transformations. Then in Eq. (28) we must also take into account that the integrand must transform as a scalar density achieved by introducing the square root of the absolute Jacobian. Thus Eq. (28) is replaced by

\[
I = \int \left[ \frac{k}{2t} \left( \frac{\partial \phi^*}{\partial t} - \frac{\partial \phi}{\partial t} \right) + \frac{k^2}{2m} \mu^\nu \frac{\partial \phi^*}{\partial x^\nu} \frac{\partial \phi}{\partial x^\mu} \right] \sqrt{\det h} d^4x.
\]

This (apart from notation) is identical to the standard procedure adopted for obtaining the Schrödinger equation in curvilinear coordinates (where \( \gamma_0 = 1 \), and \( -h^{\mu\nu} \) is the inverse of the metric tensor of 3-space). Thus, Eq. (28) is replaced by

\[
I = \int \left[ \frac{k}{2t} \left( \frac{\partial \phi^*}{\partial t} - \frac{\partial \phi}{\partial t} \right) + \frac{k^2}{2m} \mu^\nu \frac{\partial \phi^*}{\partial x^\nu} \frac{\partial \phi}{\partial x^\mu} \right] \sqrt{\det h} d^4x.
\]

As discussed in Sec. III, we consider conformal transformations as transformations on the coordinates alone, but not on the tensors \( \gamma_0 \) and \( h^{\mu\nu} \), in conformity with the usual interpretation of transformations under the group \( C_p \). Thus we have to investigate whether it is possible to maintain the form of the Hamilton–Jacobi equation as the Schrödinger equation under these conditions. Clearly, for the Hamilton–Jacobi equation this will be the case if in Eq. (27) a transformation of the coordinates and of \( S \) will yield an equation of the same form, possibly multiplied by an over-all factor. For invariance of the Schrödinger equation following from the variational principle (31), on the other hand, it is necessary that the transformation of the coordinates (including the volume element) and of \( \psi \) will leave the integrand of (31) invariant up to a divergence. It should be noted that with the interpretation adopted here the factor \( \sqrt{\det h} h^{-1/2} \) does not change under conformal transformations, but that \( d^4x \) changes to \( d^4x' \).

This interpretation has no effect on the Galilei invariance of the equations. However, it is clear that the equations are not invariant under the full group \( C_p \) which involves the general transformations (14G), or even under the transformations restricted only by the condition (16G) leading to (18Ga). We shall therefore investigate instead the possible invariance under the various subgroups.

We first consider the well-known case of subgroup I, i.e., the behavior of Eqs. (25) or (27) and (28) or (31) under Galilei transformations. Clearly, space and time translations as well as rotations leave them unchanged, with \( S \) and \( \psi \) transforming as scalars. However, for the Galilei "boosts"

\[
I' = I, \quad x'^\mu = x^\mu + \epsilon^\mu,
\]

Eq. (25) becomes

\[
\frac{\partial S}{\partial t} + \epsilon^\nu \frac{\partial S}{\partial x^\nu} = 0.
\]

This can easily be seen to be of the form (25) in the transformed quantities if we take

\[
S' = S + m \epsilon^\nu x^\nu.
\]

Applying the transformations (32) to Eq. (28), we obtain

\[
I = \int \left[ \frac{k}{2t} \left( \frac{\partial \phi^*}{\partial t} - \frac{\partial \phi}{\partial t} \right) + \frac{k^2}{2m} \mu^\nu \frac{\partial \phi^*}{\partial x^\nu} \frac{\partial \phi}{\partial x^\mu} \right] \sqrt{\det h} d^4x'.
\]

To establish the invariance of the Schrödinger equation, it is sufficient to establish the invariance of the variational principle under infinitesimal transformations. It can easily be verified that, for infinitesimal \( \epsilon' \), Eq. (35) is of the form (32) in the transformed quantities if we take

\[
\psi' = \psi (1 + \imath k \epsilon' x^\nu),
\]

which is the infinitesimal form of multiplication of \( \psi \) by a phase factor.

Now we consider the general dilatations VI. Then Eq. (25) becomes

\[
\frac{\partial S}{\partial t} + \frac{1}{2mD^2} \frac{\partial S}{\partial x^\nu} \frac{\partial S}{\partial x^\mu} = 0.
\]

This is of the same form as Eq. (25) provided that we choose

\[
S' = BD^2S,
\]

and thus the free-particle Hamilton–Jacobi equation is invariant under VI (up to a factor \( D^2B^2 \)) as well as under its subgroup IV.
For Eq. (31), the transformations VI yield
\[
I = \int \left[ \frac{\hbar}{2iB} \left( \frac{\partial \psi^*}{\partial x'} - \psi^* \frac{\partial \psi}{\partial x'} \right) \right] (1 + 2\lambda + 2\lambda') \, d^3x'.
\]
This is of the form (26) for the transformed quantities only if
\[
B = D^2, \quad \psi = D^{1/2} \psi,
\]
i.e., only for the subgroup IV (with \(D = C\)) of the transformations VI.

Now we consider the subgroup V. Then Eq. (25) becomes
\[
\frac{1}{1 - \lambda^2} \left( \frac{\partial S}{\partial t'} + D'x' \frac{\partial S}{\partial x'} + \frac{1}{2m} \frac{\partial S}{\partial x'} \frac{\partial S}{\partial x'} \right) = 0.
\]
It can easily be verified that this reduces to the form (25) apart from an irrelevant over-all factor \((1 - \lambda^2)^{-2}\) provided that we choose
\[
S' = S + \frac{mC_0 x^* x'}{2(1 + \lambda^2)}.
\]

To investigate the invariance of Eq. (31) under the subgroup V, it is simpler to consider only infinitesimal transformations, with \(D = \lambda\). Then Eq. (31) is transformed to
\[
I = \int \left[ \frac{\hbar}{2iB} \left( \frac{\partial \psi^*}{\partial x'} - \psi^* \frac{\partial \psi}{\partial x'} \right) \right] (1 + 2\lambda + 2\lambda') \, d^3x'.
\]
If \(\lambda\) is of the form (26) for the transformed quantities if we choose
\[
\psi' = \psi \left( 1 - \frac{\lambda}{2m} \frac{x^* x'}{2h} \right) = \psi \left( 1 - \frac{\lambda}{2m} - \frac{\lambda}{2m} x^* x' \right).
\]
Thus the Schrödinger equation is invariant under the Schrödinger group, and the Hamilton–Jacobi equation is invariant under a 13-parameter group, the product of the subgroups I, V, and VI. Neither equation is invariant under subgroups II or III.

Obviously, these statements remain correct if we consider \(N\) noninteracting particles instead of just one free particle. In this case, of course, there exist additional transformations that leave the equations invariant which, however, are of no interest for our discussion.

In the presence of interactions Eq. (25) is replaced by
\[
\frac{\partial S}{\partial t} + \frac{1}{2m} \frac{\partial S}{\partial x'} = \frac{1}{2m} \frac{\partial S}{\partial x'} \frac{\partial S}{\partial x'} + V = 0,
\]
and Eq. (26) by
\[
I = \int \left[ \frac{\hbar}{2iB} \left( \frac{\partial \psi^*}{\partial x'} - \psi^* \frac{\partial \psi}{\partial x'} \right) \right] (1 + 2\lambda + 2\lambda') \, d^3x',
\]
with corresponding changes in the subsequent equations. For interactions of the form
\[
V = \frac{1}{2} \sum_{i,j=1}^{N} V_{ij} (\psi_i^* \psi_j) - \frac{1}{2} \sum_{i,j=1}^{N} (\psi_i \psi_i^*) (\psi_j \psi_j^*)^{1/2},
\]
these equations are, of course, invariant under the Galilei transformations I regardless of the form of \(V_{ij}\). It can readily be verified that the Hamilton–Jacobi equation remains invariant under the transformations V and VI, and the Schrödinger equation under V and IV, however, only if
\[
V_{ij} = C_{ih} C_{ij} C_{hj}.
\]

The invariance of this particular potential was not recognized in Ref. 7 in which the name "Schrödinger group" was suggested (but was noted later by Burdet and Perrin’). On the other hand, it was known to Jacobi25 that the equations of motion of a Newtonian N-body system with interactions of the form (48) are invariant under the transformations IV and V in addition to those of the Galilei group, and therefore the Schrödinger group should more appropriately be called the Jacobi–Schrödinger group.

V. DISCUSSION

In Sec. III we briefly discussed the conformal extension \(C_\phi\) of the Poincaré group. It can be characterized by a tensor \(\tilde{\eta}_{\nu \sigma}\) related to the Minkowski metric \(\eta_{\nu \sigma}\) by Eq. (12P), its inverse \(\tilde{\eta}^{\nu \sigma}\) is given by (13P). The transformations of \(C_\phi\) are explicitly given by Eq. (11P), which has the simple Galilean limit \(C_\phi\) given by Eqs. (11G). Both \(C_\phi\) and \(C_\phi\) are 15-parameter groups.

However, we can instead define conformal extensions of the Galilei group directly. The most general conformal extension \(C_\phi\) is given by the transformations (14G), for which the tensor \(\tilde{\eta}_{\nu \sigma}\) is related to the Galilean "metric" \(\eta_{\nu \sigma}\) by Eq. (12G), which is analogous to Eq. (13P) and indeed is its Galilean limit. However, since neither \(\tilde{\eta}_{\nu \sigma}\) nor \(\tilde{\eta}^{\nu \sigma}\) possess an inverse, the analog \(\tilde{\eta}^{\nu \sigma}\) of \(\eta^{\nu \sigma}\) requires an independent definition, which is only restricted by the conformal analog of Eq. (9G). The most general \(\tilde{\eta}^{\nu \sigma}\) allowed by this satisfies Eq. (15G), which is a much less restrictive relation between \(\tilde{\eta}^{\nu \sigma}\) and \(\eta^{\nu \sigma}\) than the corresponding relation (13P) between \(\eta^{\nu \sigma}\) and \(\eta^{\nu \sigma}\). A relation more closely analogous to (13P) is Eq. (16C), which together with (12C) defines a group \(C_\phi\) of transformations given by Eqs. (14Ca) and (16C). An even closer analogy with (13P) is obtained by imposing Eq. (21G), which together with (12C) defines a group \(C_\phi\) of transformations \(C_\phi\) given by Eqs. (14Ca) and (22C). From their definitions, \(C_\phi\) is a subgroup of \(C_\phi\), which is a subgroup of \(C_\phi\). All three groups are gauge groups.

If the arbitrary functions in these groups are restricted in various ways, a number of subgroups can be obtained. The most important ones are the 15-parameter group \(C_\phi\), which is a subgroup of \(C_\phi\), but not of \(C_\phi\), and the 12-parameter Jacobi–Schrödinger group \(C_\phi\), which is a subgroup of \(C_\phi\). Both \(C_\phi\) and \(C_\phi\) contain the Galilei group as a subgroup, but otherwise they do not overlap.

In Sec. IV we established the invariance of the free-particle Schrödinger equation under \(C_\phi\) by investigating the behavior of the variational principle (26) for this equation under the infinitesimal transformations of \(C_\phi\).
Unlike other authors, 5-8 we did not have to consider the mass \( m \) as a quantity subject to transformations, nor did we have to define new transformations from those of \( C_0 \) to absorb a change of mass into the coordinate transformations (as was done in Ref. 8). We also established the invariance of the free-particle Hamilton–Jacobi equation under a 13-parameter group containing \( C_5 \).

The invariance of the variational principle (26) under a 12-parameter group, by Noether’s theorem, implies the existence of 12 local conservation laws. These will be discussed elsewhere, as will be the corresponding classical laws for the Hamilton–Jacobi equation. 26

The close correspondence between the Hamilton–Jacobi and the Schrödinger equation has, of course, been known for half a century, and our results further illustrate this correspondence. 27

The behavior of the Schrödinger equation under arbitrary accelerations, i.e., under the group \( C_2 \), is more appropriately discussed in connection with a consideration of the equivalence principle, and is the subject of a paper by J. Stachel in preparation. 23,28

The various extensions of the Galilei group considered here give rise to two-body invariants of importance in a generalized dynamics which will be discussed elsewhere 29 in connection with the two-body invariants of the Galilei group found earlier. 13,14

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2For a brief review of the conformal group and its older applications in physics see T. Fulton, F. Rohrlich, and L. Witten, Rev. Mod. Phys. 34, 442 (1962).
12P. Havas, Rev. Mod. Phys. 36, 939 (1964), Sec. V.
13Ref. 12, Sec. III.
15In Secs. II and III, corresponding formulas for the Poincaré and the Galilei group (or their extensions) will be designated by \( P \) and \( G \), respectively; formulas without a letter hold for both cases.
16These tensors were first introduced by K. Friedrichs, Math. Ann. 98, 966 (1927). A related covariant formulation of Newtonian theory was given earlier by E. Cartan, Ann. École Norm. 40, 325 (1923); 41, 1 (1924).
17In Ref. 4, \( e^{-2} \) is expressed in terms of dimensionless variables \( x^a/\lambda \) where all \( x^a \) are chosen to have dimensions of length. For our present purposes, such a representation is not convenient.
19In Ref. 12, the conditions are stated in Eqs. (1129); however, in the last determinant the fourth row and the column should have been omitted.
22In Ref. 12, the conditions are stated in Eqs. (1129); however, in the last determinant the fourth row and the column should have been omitted.
27This correspondence has recently been extended to the problem of separation of variables by P. Havas, J. Math. Phys. 16, 1961, 2476 (1973), and some of our results may be applicable to this problem, as will be discussed elsewhere.
28For discussions of the case of constant acceleration from different points of view see Ref. 8 and G. Rosen, Am. J. Phys. 40, 663 (1972).
29P. Havas and J. Plebański (to be published shortly).