Complex Relativity and Double KS Metrics.

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Summary. — The basic motivations and the foundamental facts of the theory of the double KS metrics are outlined. The complex metrics $n_{\mu\nu} + 2P_{\mu\nu} k_{\nu} + 2Q_{\mu\nu} l_{\nu}$ with $k_{\mu}$ and $l_{\mu}$ null and mutually orthogonal become particularly interesting when $k_{\mu}$ and $l_{\mu}$ are surface forming, i.e. when they determine a congruence of the complex null strings.

1. - Introduction.

The idea of obtaining new solutions of real equations by the device of studying the complex extension, i.e. the analytic continuation, of these equations has a long history in mathematical physics. In 1887 (1), Paul Appel derived from the basic real solution $\varphi = m(x^2 + y^2 + z^2)^{-\frac{1}{4}}$ of Laplace's equation $\Delta \varphi = 0$ a new solution by replacing $x, y, z$ by $x + i\alpha, y + i\beta, z + iy$ and $m$ by $\mu + i\lambda$, and then taking the real part. The new solution is singular along a circle which basically coincides with the Kerr circle, the singularity of the Kerr-Newman solution. The classical textbook of Courant and Hilbert (2) outlines basic tech-

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niques which provide new solutions by the use of the complex extension in the case of linear partial differential equations. The works of Robinson and Trautman (4) apply these techniques to the relativistic Maxwell equations. The recent work of Newman and his co-workers (4) and the study by Newman and Penrose (5) and others (6) of «heaven» has shown the importance of the analytic extension of relativistic theories. The idea of working with such an extension of the field equations of general relativity is one of the basic ingredients of this paper.

The second basic ingredient is the study of metrical structures of the KS type (6,10), i.e. metrics of the form

\[ g_{\mu\nu} = \eta_{\mu\nu} + 2\hbar k_{\mu} k_{\nu}, \quad \eta^{\mu\nu} k_{\mu} k_{\nu} = 0, \tag{1.1} \]

where \(\eta_{\mu\nu}\) is a flat, nonsingular metric. This has a shorter history. Eddington (11) knew already that the Schwarzschild solution can be written in this form. The paper by DeBney, Kerr and Schild (12) shows that the Kerr-Newman metric can also be written in this form, as well as many other solutions of the Einstein-Maxwell equations. However, the generalization of the Kerr-Newman metric to the case of a nonzero cosmological constant does not seem to be possible for metrics of the form (1.1), which are too restrictive.

Recently, Demiański and Plebański (13-18) gave a general class of solutions of the Einstein-Maxwell equations, which are of type \(D\), and which can be written in the double KS form

\[ g_{\mu\nu} = \eta_{\mu\nu} + \mathcal{K} k_{\mu} k_{\nu} + \mathcal{L} l_{\mu} l_{\nu}, \tag{1.2} \]

where $\eta_{\mu\nu}$ is the flat, nonsingular metric, and where

\begin{equation}
\eta^\mu\nu k_\mu k_\nu = \eta^\mu\nu l_\mu l_\nu = \eta^\mu\nu k_\mu l_\nu = 0.
\end{equation}

This metric is necessarily complex for $k_\mu$ and $l_\mu$ linearly independent, since in a real $V_4$ with signature $(+++-)$ two null vectors which are orthogonal must be proportional.

The solution is given in the form (1.2) with

\begin{equation}
k_\mu \, dx^\mu = du + q^2 \, dv \quad \text{and} \quad l_\mu \, dx^\mu = du - p^2 \, dv,
\end{equation}

where $\{u, v, p, q\}$ are co-ordinates. The flat metric is

\begin{equation}
dl^2 = \eta_{\mu\nu} \, dx^\mu \, dx^\nu = 2 \, dw^1 \, dw^2 + 2 \, dw^3 \, dw^4,
\end{equation}

where \{\omega^1, \omega^2, \omega^3, \omega^4\} are co-ordinates. The electromagnetic field is given by

\begin{equation} \begin{split}
\omega^1 &= \frac{u - ipqv}{1 - pq}, \\
\omega^2 &= \frac{q + ip}{1 - pq}, \\
\omega^3 &= -\frac{q + ip}{1 - pq} (u - iv), \\
\omega^4 &= \frac{1}{1 - pq}.
\end{split} \end{equation}

The physical interpretation of the 7 constants in $P$ and $Q$ is as follows:

\begin{itemize}
\item $\lambda$ is the cosmological constant,
\item $m$ the mass,
\item $n$ a magnetic mass or NUT parameter,
\item $e$ the charge,
\item $g$ the magnetic charge;
\item $\varepsilon$ and $\gamma$ are related to the Kerr and Levi-Civita parameters which describe the uniform rotation and acceleration of the sources.
\end{itemize}

The metric and electromagnetic field described above, fulfill complex Einstein-Maxwell equations, with all parameters and co-ordinates considered complex; notice that under Hodge's star $\ast \omega = \omega$, $\ast \bar{\omega} = -\bar{\omega}$ so that both differentials and co-differentials of $\omega$ and $\bar{\omega}$ vanish.
Real solutions arise if in place of $u$ and $v$ we introduce

$$
\begin{align*}
\tau := u - \int \frac{q^2 \, dq}{Q} + i \int \frac{p^2 \, dp}{P}, \\
\sigma := v + \int \frac{dq}{Q} + i \int \frac{dp}{P},
\end{align*}
$$

(1.7)

and we consider now all constants and co-ordinates $\{\tau, \sigma, p, q\}$ as real; the form $\bar{\omega}$ then becomes the complex conjugate of $\omega$.

The result outlined above suggests that it is of interest to consider in complex $V_4$'s the general metrics of the type (1.2), understanding (1.1) as the special case in which the two mutually orthogonal vectors are linearly dependent.

Below, we outline some basic facts related to the theory of the complex double KS metrics.

2. - DKS conjugation, algebraic properties.

Let $V_4$ be a complex Riemannian space, i.e. a pair consisting of i) an analytic (complex) differential variety $M_4$ which carries the natural structure of the (complex) exterior-form algebra $A = \bigoplus_{p=0}^4 A^p$ and ii) $ds^2 \in A^1 \otimes A^1$, i.e. the Riemannian (complex) metric defined in the tensorial product of $A^1$ by itself. Vectors are elements of $A^1$; having the metric, we also have duality operation, Hodge's star which we choose to normalize so that, for every $\omega \in A^p$, $** \omega = \omega$.

Then, with $u, v \in A^1$ we define the scalar product of the two vectors as the mapping $A^1 \times A^1 \to \mathbb{C}$ determined by $u \langle v = \langle u \land \ast v \rangle$.

Now let $N$ be a subspace of $A^1$ of dim $(N) = 2$ endowed with the property

$$
\tag{2.1} u, v \in N \Rightarrow \langle u \vert v \rangle = 0.
$$

Thus $N$ is to be understood as a set of 1-forms spanned by a pair of mutually orthogonal—and, of course, linearly independent—null vectors. If $k, l \in A^1$ form a base of $N$, then

$$
\tag{2.2}
\begin{align*}
i) \quad & k \lvert k = k \lvert l = l \lvert l = 0, \\
ii) \quad & \alpha, \beta \in \mathbb{C}, \quad \alpha k + \beta l = 0 \Rightarrow \alpha = 0 = \beta.
\end{align*}
$$

(Of course, given $N$, the base is given modulo $GL(2, \mathbb{C})$ transformations only.)

We can now define the space $\bar{V}_4$, KS conjugated to $V_4$ as the pair consisting of i) the same $M_4$ which enters in the definition of $V_4$ and ii) the (complex) Riemannian metric defined by $d\bar{s}^2 \in A^1 \otimes A^1$ which is endowed with the prop-
The construction outlined above is evidently invariant and independent of the use of some local co-ordinates. It is, however, useful to elucidate the meaning of this construction by working in some co-ordinate patch of $M_4, \{x^\mu\}$. Let the base of $N$ be represented by $k = k_\mu dx^\mu$, $l = l_\mu dx^\mu$ and let the metrics of $V_4$ and $\bar{V}_4$ be respectively $g_{\mu\nu}$ and $\bar{g}_{\mu\nu}$. Then, assumption (2.2) is equivalent to

\begin{equation}
(2.4) \begin{cases}
i) \quad g^{\mu\nu} k_\mu k_\nu = g^{\mu\nu} l_\mu l_\nu = g^{\mu\nu} l_\mu k_\nu = 0, \\
ii) \quad \alpha, \beta \in \mathbb{C}, \quad \alpha k_\mu + \beta l_\mu = 0 \rightarrow \alpha = 0 = \beta,
\end{cases}
\end{equation}

and $N \ni v_\mu = \alpha k_\mu + \beta l_\mu; \alpha, \beta \in \mathbb{C}$. The condition (2.3) takes now the form

\begin{equation}
(2.5) \quad \bar{g}_{\mu\nu} = g_{\mu\nu} + 2P k_\mu k_\nu + 2Q l_\mu l_\nu + 4R k_\mu l_\nu,
\end{equation}

where $P, Q$ and $R$ are scalars.

Now one easily establishes that (2.4i) and (2.5) algebraically imply

\begin{equation}
(2.6) \quad \det(\bar{g}_{\mu\nu}) = \det(g_{\mu\nu})
\end{equation}

and that

\begin{equation}
(2.7) \quad \bar{g}_{\mu\nu} k_\mu k_\nu = \bar{g}_{\mu\nu} l_\mu l_\nu = \bar{g}_{\mu\nu} k_\mu l_\nu = 0.
\end{equation}

From this we infer that if $V_4$ is either nonsingular or singular, the conjugated space $\bar{V}_4$ inherits the corresponding property. Moreover, the scalar products of the vector forms $N$ also vanish in the sense of the space $\bar{V}_4$. From this one sees that, if $\bar{V}_4$ is KS conjugated to $V_4$, then $V_4$ is also KS conjugated to $\bar{V}_4$. Thus, the relation of the KS conjugation, $\sim$, is reflexive (but, in general, not transitive).

It can be observed that (2.4i) and (2.5) imply

\begin{equation}
(2.8) \quad \bar{g}_{\mu\nu} = g_{\mu\nu} - 2P k_\mu k_\nu - 2Q l_\mu l_\nu - 4R k_\mu l_\nu,
\end{equation}

where the contravariant components of the null vectors are obtained by manipulating indices by using equivalently either $g_{\mu\nu}$ or $\bar{g}_{\mu\nu}$:

\begin{equation}
(2.9) \quad k_\mu = g_{\mu\nu} k_\nu \equiv \bar{g}_{\mu\nu} k_\nu, \quad l_\mu = g_{\mu\nu} l_\nu \equiv \bar{g}_{\mu\nu} l_\nu.
\end{equation}

With $k_\mu$ and $l_\mu$ meaningful modulo the transformations

\begin{equation}
(2.10) \quad GL(2, \mathbb{C}) \begin{cases}
k_\mu = \alpha k_\mu^' + \beta l_\mu^', \\
l_\mu = \gamma k_\mu^' + \delta l_\mu^',
\end{cases} \quad \alpha \delta - \beta \gamma \neq 0,
\end{equation}
the scalars $P$, $Q$ and $R$ in (2.5) do not possess individual invariant meaning. However, the discriminant

\begin{equation}
\Delta := PQ - R^2
\end{equation}

is a $GL(2, \mathbb{C})$ density. Therefore, it makes an invariant sense to distinguish within the general concept of KS conjugation the two subcases: i) the single KS conjugation, denoted subsequently SKS, which occurs for $\Delta = 0$ and ii) the double KS conjugation, denoted subsequently DKS, which corresponds to the case when $\Delta \neq 0$.

It is obvious that when $\tilde{\mathcal{V}}_4$ is SKS conjugated to $\mathcal{V}_4$, then with no loss of generality we can replace (2.5) by

\begin{equation}
\tilde{g}_{\mu\nu} = g_{\mu\nu} + 2hm_\mu m_\nu,
\end{equation}

where

\begin{equation}
g^{\mu\nu}m_\mu m_\nu = 0,
\end{equation}

i.e. with $m = m_\mu dx^\mu \in \mathcal{N}$. In the case of $\tilde{\mathcal{V}}_4$ DSK conjugated to $\mathcal{V}_4$ we can select the $GL(2, \mathbb{C})$ gauge so that

\begin{equation}
\tilde{g}_{\mu\nu} = g_{\mu\nu} + 4Rk_\mu l_\nu,
\end{equation}

or, if one prefers to work without the cross-term, one can assume that

\begin{equation}
\tilde{g}_{\mu\nu} = g_{\mu\nu} + 2P_k_\mu k_\nu + 2Q_l_\mu l_\nu.
\end{equation}

We can now define the single and double KS metrics sensu strictu: we will say that $\tilde{\mathcal{V}}_4$ exhibits SKS structure (i.e. has the single KS metric), if $\tilde{\mathcal{V}}_4$ is SKS conjugate of the flat space $\mathcal{V}_4$.

Similarly, we will say that $\tilde{\mathcal{V}}_4$ exhibits DKS structure (i.e. has the double KS metric), if it is DKS conjugate of the flat space.

Notice that an interesting generalization of these structures is represented by $\tilde{\mathcal{V}}_4$'s which are SKS or DKS conjugated to the conformally flat spaces.

3. – DKS conjugation, connections.

We select in $\mathcal{V}_4$ a null tetrad such that

\begin{equation}
\mathcal{V}_4: \quad ds^2 = 2e^1 \otimes e^2 + 2e^3 \otimes e^4, \quad e^a \in \Lambda^1
\end{equation}

with the inverse $\tilde{\mathcal{g}}_{\ast}$. (If $\mathcal{V}_4$ were real and of signature $(++--)$, $e^a$ would be restricted by $(e^1)^* = e^2$ with $e^3, e^4$ being real; in a complex $\mathcal{V}_4$, $e^a$ is just an arbitrary complex tetrad.)
Now, if $\mathcal{V}_4$ is DKS conjugated to $V_4$, it is natural to select the null tetrad so that $e^1$ and $e^3$ form a base of $N$, i.e. $e^1 = k_\mu dx^\mu$, $e^3 = l_\mu dx^\mu$.

Then we can represent $\mathcal{V}_4$ in the form

$$\mathcal{V}_4: \quad ds^2 = 2e^1 \otimes e^3 + 2e^3 \otimes e^1,$$

where

$$\begin{align*}
e^1 &= e^1 \\
e^2 &= e^2 + Pe^1 + Re^3 \\
e^3 &= e^3 \\
e^4 &= e^4 + Re^3 + Qe^3
\end{align*}$$

$$(3.2)$$

At the same time one easily finds that

$$\begin{align*}
\partial_1 &= \partial_1 + P\partial_2 + R\partial_4 \\
\partial_2 &= \partial_2 \\
\partial_3 &= \partial_3 + R\partial_2 + Q\partial_4 \\
\partial_4 &= \partial_4.
\end{align*}$$

$$(3.3)$$

The connections in both spaces $V_4$ and $\mathcal{V}_4$ are defined by the first structure equations

$$\begin{align*}
de^a &= e^b \wedge \Gamma^a_{\ b} , \quad \Gamma_{ab} = \Gamma_{(ab)} , \\
de^\tilde{a} &= e^\tilde{b} \wedge \Gamma^\tilde{a}_{\ \tilde{b}} , \quad \Gamma_{\tilde{a}\tilde{b}} = \Gamma_{(\tilde{a}\tilde{b})} .
\end{align*}$$

$$(3.4)$$

Tetral indices are manipulated by the numerical metric

$$\begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{pmatrix} = (g_{ab}) .$$

$$(3.5)$$

Connection forms define the Ricci-rotation coefficients

$$\Gamma_{\ ab} = \Gamma_{abc} e^c , \quad \Gamma_{\tilde{a}\tilde{b}} = \Gamma_{abc} e^{\tilde{c}} .$$

$$(3.6)$$

Notice that in the case of $\Gamma_{ab}$ we use $e^a$ as the base defining $\Gamma_{abc}$ and for $\Gamma_{\tilde{a}\tilde{b}}$ we use, respectively, $e^{\tilde{c}}$ as the base which determines $\Gamma_{abc}$. The easiest technique which leads to relationships among $\Gamma$'s of both types consists in computing
them from the commutation rules (*)

\[ \partial_i \partial_a - \partial_a \partial_i = (\Gamma^{ab}_{c} - \Gamma^{ba}_{c}) \partial_c, \quad \partial_i \partial_a - \partial_a \partial_i = (\Gamma^{ab}_{c} - \Gamma^{ba}_{c}) \partial_c. \]

The complete table of these relationships is given below.

\[ \Gamma_g^{ab} = \Gamma^{ab}_{g} = \Gamma^{ab}_{g}, \quad \Gamma_{g^2} = \Gamma^{ab}_{g}, \quad \Gamma_{g^3} = \Gamma^{ab}_{g}, \quad \Gamma_{g^4} = \Gamma^{ab}_{g}. \]

\[ \Gamma_{g^5} = \Gamma_{g^5} = \Gamma_{g^5} = \Gamma_{g^5}, \quad \Gamma_{g^6} = \Gamma_{g^6} - \Gamma_{g^6}. \]

\[ \Gamma_{g^7} = \Gamma_{g^7} - \Gamma_{g^7} + \Gamma_{g^7} - \Gamma_{g^7}, \quad \Gamma_{g^8} = \Gamma_{g^8} - \Gamma_{g^8}. \]

\[ \Gamma_{g^9} = \Gamma_{g^9} - \Gamma_{g^9} + \Gamma_{g^9} - \Gamma_{g^9}, \quad \Gamma_{g^{10}} = \Gamma_{g^{10}} - \Gamma_{g^{10}} + \Gamma_{g^{10}} - \Gamma_{g^{10}}. \]

\[ \Gamma_{g^{11}} = \Gamma_{g^{11}} - \Gamma_{g^{11}} + \Gamma_{g^{11}} - \Gamma_{g^{11}}, \quad \Gamma_{g^{12}} = \Gamma_{g^{12}} - \Gamma_{g^{12}} + \Gamma_{g^{12}} - \Gamma_{g^{12}}. \]

\[ \Gamma_{g^{13}} = \Gamma_{g^{13}} - \Gamma_{g^{13}} + \Gamma_{g^{13}} - \Gamma_{g^{13}}, \quad \Gamma_{g^{14}} = \Gamma_{g^{14}} - \Gamma_{g^{14}} + \Gamma_{g^{14}} - \Gamma_{g^{14}}. \]

\[ \Gamma_{g^{15}} = \Gamma_{g^{15}} - \Gamma_{g^{15}} + \Gamma_{g^{15}} - \Gamma_{g^{15}}, \quad \Gamma_{g^{16}} = \Gamma_{g^{16}} - \Gamma_{g^{16}} + \Gamma_{g^{16}} - \Gamma_{g^{16}}. \]

(*) A suggestion of Dr. J. D. Finley related to this point is appreciated.
Notice that the right-hand members of (3.19) i) contain the derivatives \( \dd_\sigma \) (and not simply \( \partial_\sigma \)) and ii) they contain terms \( -\Gamma_{\bar{1}33} + \Gamma_{\bar{1}3\bar{1}} \) and \( \Gamma_{\bar{4}73} + \Gamma_{\bar{4}7\bar{1}} \) which can be expressed in terms of \( I^n \)'s by using the previous formulae.

Although in applications DKS metrics are likely to be of the simpler form (2.14) or (2.15), we decided to use an arbitrary gauge for \( P, Q \) and \( R \) for reasons of completeness. By specializing our list one can find the needed answer in all practically useful cases.

In particular, by formally setting in our formulae \( P = 0 = R, Q = :h, \) we obtain the transformation rules of connections for the two SKS conjugated metrics.

\[
\begin{align*}
(3.20) & \quad \Gamma_{\bar{4}71} = \Gamma_{421}, \quad \Gamma_{\bar{4}73} = \Gamma_{423}, \quad \Gamma_{\bar{4}75} = \Gamma_{425}, \quad \Gamma_{\bar{4}77} = \Gamma_{427}. \\
(3.21) & \quad \Gamma_{417} = \Gamma_{412}, \quad \Gamma_{411} = \Gamma_{411}, \quad \Gamma_{415} = \Gamma_{415}, \quad \Gamma_{417} = \Gamma_{417}. \\
(3.22) & \quad \Gamma_{\bar{3}71} = \Gamma_{321} - Q\Gamma_{421}, \quad \Gamma_{\bar{3}73} = \Gamma_{323} - Q\Gamma_{423}, \\
& \quad \Gamma_{\bar{3}75} = \Gamma_{325} + Q_{,2} + Q(2\Gamma_{342} + \Gamma_{422} - \Gamma_{374}) - Q^2\Gamma_{424}, \\
& \quad \Gamma_{\bar{3}77} = \Gamma_{327}. \\
(3.23) & \quad \Gamma_{\bar{3}72} = \Gamma_{322} - Q\Gamma_{422}, \quad \Gamma_{\bar{3}74} = \Gamma_{324} - Q\Gamma_{424}, \\
& \quad \Gamma_{\bar{3}76} = \Gamma_{326} + Q_{,4} + Q(\Gamma_{412} + 2\Gamma_{342} - \Gamma_{314}) - Q^2\Gamma_{414}, \\
& \quad \Gamma_{\bar{3}78} = \Gamma_{328}. \\
(3.24) & \quad \Gamma_{1\bar{7}1} = \Gamma_{111}, \quad \Gamma_{1\bar{7}3} = \Gamma_{113}, \quad \Gamma_{1\bar{7}5} = \Gamma_{115} + Q(\Gamma_{431} - \Gamma_{413} - \Gamma_{134}), \\
& \quad \Gamma_{1\bar{7}7} = \Gamma_{117}. \\
(3.25) & \quad \Gamma_{3\bar{7}1} = \Gamma_{341} - Q\Gamma_{414}, \quad \Gamma_{3\bar{7}3} = \Gamma_{343} - Q\Gamma_{414}, \\
& \quad \Gamma_{3\bar{7}5} = \Gamma_{345} + Q_{,4} + Q\Gamma_{444}, \quad \Gamma_{3\bar{7}7} = \Gamma_{347}.
\end{align*}
\]

Particularly interesting are the SKS conjugated metrics where the distinguished direction \( e^3 \) is geodesic and shear free in the sense of \( V_4 \), i.e.

\[
(3.26) \quad \begin{cases} 
\Gamma_{4\bar{1}4} = 0, \\
\Gamma_{413} = 0, \\
\Gamma_{414} = 0, \\
\Gamma_{411} = 0.
\end{cases}
\]

(Of course, we talk now about the real case where \( e^4 = (e^4)^* \) and \( e^3, e^4 \) are real.) Then, from (3.20) and (3.21) we infer that \( e^3 \) is also geodesic and shear free in the sense of \( V_4 \). Moreover, one easily sees that \( \Gamma_{\bar{a}b\sigma} = \Gamma_{a\sigma b} \) except for
the following components:

\[
\begin{align*}
\gamma_{31} &= \Gamma_{31} - Q\Gamma_{41}, \\
\gamma_{21} &= \Gamma_{21} + Q(I_{42} + 2\Gamma_{41} - \Gamma_{31}), \\
\gamma_{12} &= \Gamma_{12} - Q(I_{42} - \Gamma_{41} - \Gamma_{31}), \\
\gamma_{13} &= \Gamma_{13} + Q(I_{42} - 2\Gamma_{41} + \Gamma_{31}).
\end{align*}
\]  

(3.27)

Thus, in this case, SKS conjugation transforms 6 components of \( \Gamma_{abc} \) keeping the remaining 18 invariant.

Returning now to the general case, we should like to suggest that particularly interesting are DKS conjugated spaces with the hereditary property (see (3.9)) that

\[
\Gamma_{444} = 0 = \Gamma_{442}.
\]  

(3.28)

From the work of (1) we know the interpretation of these conditions: they mean that the null directions \( e^1 \) and \( e^3 \) are surface forming; the congruence of surfaces with the surface elements \( e^1 \wedge e^3 \) has the important property that the cotangent space spanned by \( e^1 \) and \( e^3 \) is parallelly propagated along the surfaces of our family, i.e.

\[
u, v \in N \rightarrow \xi v = u^\mu v^\kappa, u_\kappa \in N.
\]  

(3.29)

The corresponding 2-dimensional null varieties are thus geodesic surfaces; we will subsequently refer to them as null strings, as was proposed in (7)\(^(*)\).

If, therefore, \( N \) represents a congruence of tangent directions to null strings, i.e. if the conditions (3.28) apply, then the formulae (3.9)-(3.19) significantly simplify: \( e^1 \) and \( e^3 \) are also surface forming in \( \mathcal{V}_4 (\Gamma_{444} = 0 = \Gamma_{442}) \) and we have the equality \( \Gamma_{a^cb^d}^c = \Gamma_{a^bc^d} \) for all combinations of indices where at least two out of \( (abc) \) take values 4 or 2 (with possibility of repetitions). There are twelve (out of 24) such \( \Gamma \)'s.

We should also recall that according to the generalized Goldberg-Sachs theorem form (7), \( N \) represents a congruence of tangents to null strings if and only if—assuming that \( V_4 \) is Einstein flat—the conformal curvature is algebraically degenerate.

4. — Some vanishing components.

Let \( V_4 \) be fiat and let \( \mathcal{V}_4 \) be DKS conjugated to \( V_4 \) with \( e^1 \) and \( e^2 \) surface forming, so that

\[
\Gamma_{444} = \Gamma_{442} = \Gamma_{424} = \Gamma_{444} = 0.
\]  

(4.1)

\(^(*)\) The use of the term "null strings" is different here from that of (18), though, in fact, it describes a subcase with all vectors complex null and orthogonal.

Using eqs. (4.1) and (3.3) in the table of connections, eqs. (3.9) to (3.17), we find that

\begin{equation}
\Gamma_{43}^{1} = \Gamma_{42} = \Gamma_{41}^{1} e^{1} + \Gamma_{433}^{3} e^{3}
\end{equation}

and

\begin{equation}
\Gamma_{43}^{3} - \Gamma_{42}^{3} = \{P_{2} + 2P \Gamma_{123} + R(\Gamma_{21} - 2 \Gamma_{41})\} e^{1} + \\
+ \{R_{3} + P \Gamma_{333} + R(\Gamma_{12} - \Gamma_{34} + \Gamma_{43}) - Q(\Gamma_{43} - \Gamma_{43})\} e^{3},
\end{equation}

\begin{equation}
\Gamma_{43}^{4} - \Gamma_{34}^{4} = \{R_{4} + P(\Gamma_{34} - \Gamma_{43}) + R(\Gamma_{44} + \Gamma_{34}) - Q \Gamma_{44}\} e^{4} + \\
+ \{Q_{4} + 2Q \Gamma_{44} + R(\Gamma_{34} - \Gamma_{43})\} e^{8}.
\end{equation}

Now, the curvature form \( \mathcal{K}_{ab} = R_{a\beta \gamma \delta} e^{\beta} \wedge e^{\delta} \) of \( V_{a} \) and \( \mathcal{K}_{\alpha \beta} = R_{\alpha \beta \delta \epsilon} e^{\delta} \wedge e^{\epsilon} \) of \( V_{\alpha} \) are given by the second structure equations

\begin{equation}
\begin{cases}
\frac{1}{2} \mathcal{K}_{\alpha \beta} := d \mathcal{K}_{\alpha \beta} + \mathcal{K}_{\alpha \beta} \wedge \mathcal{K}_{\alpha \beta} = 0, \\
\frac{1}{2} \mathcal{K}_{\alpha \beta} := d \mathcal{K}_{\alpha \beta} + \mathcal{K}_{\alpha \beta} \wedge \mathcal{K}_{\alpha \beta} = 0,
\end{cases}
\end{equation}

\( \mathcal{K}_{\alpha \beta} \) being zero since \( V_{a} \) is flat. Specializing these equations for \( (ab) = (42) \) and substracting, using in doing so (4.2), we obtain

\begin{equation}
\frac{1}{2} \mathcal{R}_{43} = \mathcal{K}_{42} = (\Gamma_{12}^{4} + \Gamma_{54}^{4} - \Gamma_{41}^{4} - \Gamma_{34}^{4}) e^{3},
\end{equation}

so that according to (4.2), (4.3) and (4.4) this form is proportional to \( e^{3} \wedge e^{1} \equiv e^{3} \wedge e^{1} \). On the other hand, by using the general formalism of (19) we get

\begin{equation}
\mathcal{R}_{43} = \frac{1}{2} \mathcal{C}_{(4)} e^{3} \wedge e^{5} + \frac{1}{2} \mathcal{C}_{(4)} (e^{3} \wedge e^{3} + e^{5} \wedge e^{5}) + \left( \frac{1}{2} \mathcal{C}_{(3)} - \bar{R}/12 \right) e^{3} \wedge e^{1} + \\
+ \frac{1}{2} R_{a^{4}}^{4} e^{3} \wedge e^{3} - \frac{1}{2} R_{a^{5}}^{5} (- e^{3} \wedge e^{5} + e^{5} \wedge e^{3}) - \frac{1}{2} R_{a^{5}}^{5} e^{3} \wedge e^{5},
\end{equation}

where \( R_{a^{5}}^{5} \) are the tetrad components of the Ricci tensor of \( V_{a} \), \( \bar{R} \) is its scalar curvature and \( \mathcal{C}_{(a)} \), \( a = i, \ldots, 5 \), characterize the self-dual part of the conformal curvature of \( \hat{V}_{a} \) in the standard manner (compare (18)).

Thus it follows that

\begin{equation}
R_{a^{4}}^{4} = R_{a^{5}}^{5} = R_{a^{5}}^{5} = 0
\end{equation}

and

\begin{equation}
\mathcal{C}_{(a)} = 0 = \mathcal{C}_{(4)}
\end{equation}

as the result of the assumption that \( \hat{V}_{a} \) is DKS conjugated to the flat space with \( N \) spanned by surface-forming orthogonal null vectors. We can inter-
pret (4.8) by saying that 3 out of 10 Einstein vacuum equations are automatically satisfied by such a \( \mathcal{V}_4 \). On the other hand, (4.9), according to the work of (6.7), means that the «heavenly» part of the conformal curvature of \( \mathcal{V}_4 \) is then automatically algebraically degenerated.

5. – A general flat tetrad and its connections.

Now suppose that the complex \( V_4 \) is flat. Therefore, there exist Cartesian co-ordinates \( \{x^\mu\} \) such that

\[
\begin{align*}
V_4 : & \quad ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu, \\
\|\eta_{\mu\nu}\| & = \|\text{diag} (1, 1, 1, -1)\| = \|\eta^{\mu\nu}\|.
\end{align*}
\]

Parallelly with \( x^\mu \) we will use the «null co-ordinates»

\[
\begin{align*}
\zeta & = \frac{1}{\sqrt{2}} (x^1 + ix^2), \quad u = \frac{1}{\sqrt{2}} (x^3 + x^4), \\
\xi & = \frac{1}{\sqrt{2}} (x^1 - ix^2), \quad v = \frac{1}{\sqrt{2}} (x^3 - x^4).
\end{align*}
\]

(In the present context, bar just distinguishes two different complex co-ordinates and does not imply the complex conjugation.)

Now consider the (Cartesian) Pauli matrices

\[
\|g^{AB}\| = \left( \begin{array}{ccc}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{array} \right)
\]

which fulfill the basic identities

\[
-\frac{1}{2} \eta^{\mu\nu} g^{\alpha\beta}_\mu g_{\nu}^{\alpha\beta} = \varepsilon^{\alpha\beta} \varepsilon^{\gamma\delta}
\]

and

\[
-\frac{1}{2} \varepsilon_{\alpha\beta} \varepsilon_{\gamma\delta} g^{\alpha\gamma} g_{\beta}^{\delta} = \eta_{\mu\nu}.
\]

(Of course, the spinorial Levi-Civita symbols are

\[
\|\varepsilon_{AB}\| = \|\varepsilon_{A\bar{B}}\| := \left\| \begin{array}{cc}
0 & 1 \\
1 & 0
\end{array} \right\| = \|\varepsilon^{A\bar{B}}\| = \|\varepsilon^{AB}\|.
\]

Now, with \( v^\mu \) being a complex vector, we assign to it the \( 2 \times 2 \) complex matrix

\[
v^{A\bar{B}} := \frac{1}{\sqrt{2}} v^\mu g^{A\bar{B}} \rightarrow \det [v^{A\bar{B}}] = \frac{1}{2} v^\mu v_\mu.
\]
Thus, if $v^\mu$ is null, $[v^\mu v_\nu]$ has vanishing determinant and, being $2 \times 2$ matrix, has rows proportional to columns, so that it has the dyadic form

\begin{equation}
 v^{\hat{a}\hat{b}} = k^\hat{a} m^{\hat{b}}.
\end{equation}

Consequently, a general null vector can always be represented in the form

\begin{equation}
 v^\mu = \frac{1}{\sqrt{2}} g^{\mu\hat{a}\hat{b}} k^\hat{a} m^{\hat{b}},
\end{equation}

where $k^\hat{a}$ and $m^{\hat{b}}$ are some general spinors. If now another null vector $u^\mu$ has the representation

\begin{equation}
 u^\mu = \frac{1}{\sqrt{2}} g^{\mu\hat{a}\hat{b}} l^\hat{a} n^{\hat{b}},
\end{equation}

then, for the scalar product, we have

\begin{equation}
 u^\mu v^\mu = - (k^\hat{a} l^\hat{b})(m^{\hat{a}} n^{\hat{b}}).
\end{equation}

Therefore, if $u^\mu v^\mu = 0$ and $u^\mu, v^\mu$ are linearly independent, then either $k^\hat{a} \sim l^\hat{a}$, while $m^{\hat{a}} n^{\hat{b}} \neq 0$, or $k^\hat{a} l^\hat{b} \neq 0$, while $m^{\hat{a}} n^{\hat{b}}$. 

Guided by these algebraic facts, one easily concludes that the most general null tetrad $e^a$ in a complex flat $\mathbb{C}^4$ can be constructed as follows: let $k^\hat{a}, l^\hat{a}$ and $m^{\hat{a}}, n^{\hat{b}}$ be two pairs of spinors normalized according to

\begin{equation}
 k^\hat{a} l^\hat{b} = 1 = m^{\hat{a}} n^{\hat{b}},
\end{equation}

and understood as arbitrary (sufficiently smooth) functions of the complex co-ordinates $\{x^a\}$. Then the null tetrad

\begin{equation}
 e^a := \frac{1}{\sqrt{2}} d\omega^\mu g^{\mu\hat{a}\hat{b}} \{k^\hat{a} n^{\hat{b}}, l^\hat{a} m^{\hat{b}}, - k^\hat{a} m^{\hat{b}}, l^\hat{a} n^{\hat{b}}\}
\end{equation}

is determined by some $4 \times 2 - 2 = 6$ independent complex functions.

The normalization (5.12) implies

\begin{align}
 (5.14a) & \quad k^\hat{a} l^\hat{b} - l^\hat{a} k^\hat{b} = \varepsilon_{\hat{a}\hat{b}}, \\
 (5.14b) & \quad m^{\hat{a}} n^{\hat{b}} - n^{\hat{a}} m^{\hat{b}} = \varepsilon_{\hat{a}\hat{b}}.
\end{align}

Contracting eq. (5.5) with $d\omega^\mu d\omega^\nu$ and these expressions for the $e^a$s, we find

\begin{equation}
 V_4 : \; ds^2 = 2e^1 e^3 + 2e^2 e^4
\end{equation}
with $e^a$'s given by (5.13). One also easily sees that the only scalar products among the vectors $e^a$, which are different form zero are

$$\eta^{\mu\nu} e^a_\mu e^a_\nu = 1 = \eta^{\mu\nu} e^a_\mu e^b_\nu.$$  

By taking the external differential of (5.13), we have

$$d e^a = - \frac{1}{\sqrt{2}} d x^\mu g_{\mu \nu}^{\;\;\;\nu} \wedge \begin{vmatrix} dk^a n^b \pm k_b d n^a \\
= & dl^a m^b + l^a d m^b, \\
= & -dl^a m^b - k^a d m^b, \\
= & dl^a n^b + l^a d n^b. \end{vmatrix}$$  

But (5.14) contracted with the differentials of our spinets yields

$$d k^a = k^a l^b d k^b - l^a k^b d l^b,$$
$$d l^a = k^a l^b d l^b - l^a k^b d l^b,$$
$$d m^b = m^b n^a d m^a - n^b m^a d m^a,$$
$$d n^b = m^b n^a d n^a - n^b m^a d n^a.$$

At the same time, differentiating (5.12) we have

$$k^a d l^a = l^a d k^a, \quad m^a d n^a = n^a d m^a.$$  

Substituting (5.18) in (5.17), using (5.19) and ordering, we obtain

$$d e^1 = - e^1 \wedge (l^a d k^a - n^a d m^a) + e^2 \wedge n^a d n^a + e^4 \wedge k^a d k^a,$$
$$d e^2 = e^2 \wedge (l^a d k^a - n^a d m^a) + e^3 \wedge l^a d l^a + e^4 \wedge m^a d m^a,$$
$$d e^3 = - e^1 \wedge m^a d m^a - e^2 \wedge k^a d k^a + e^3 \wedge (- l^a d k^a - n^a d m^a),$$
$$d e^4 = - e^1 \wedge l^a d l^a - e^2 \wedge n^a d n^a - e^3 \wedge (- l^a d k^a - n^a d m^a).$$

This can be compared with $d e^a = e^a \wedge \Gamma^a_\mu$ which, written in explicit form, is

$$d e^1 = - e^1 \wedge \Gamma_{12} - e^3 \wedge \Gamma_{32} - e^4 \wedge \Gamma_{42},$$
$$d e^2 = e^2 \wedge \Gamma_{12} - e^3 \wedge \Gamma_{31} - e^4 \wedge \Gamma_{41},$$
$$d e^3 = e^1 \wedge \Gamma_{41} + e^2 \wedge \Gamma_{42} - e^3 \wedge \Gamma_{34},$$
$$d e^4 = e^1 \wedge \Gamma_{31} + e^2 \wedge \Gamma_{32} + e^4 \wedge \Gamma_{44}.$$
Consequently,

\[
\begin{align*}
\Gamma_{12} &= \ell_{s} dk^{s} - n_{s} dm^{s}, & \Gamma_{34} &= \ell_{s} dk^{s} + n_{s} dm^{s}, \\
\Gamma_{43} &= -k_{s} dk^{s}, & \Gamma_{41} &= -m_{s} dm^{s}, \\
\Gamma_{31} &= -l_{s} dl^{s}, & \Gamma_{32} &= -n_{s} dn^{s},
\end{align*}
\]

(5.22)

or, combining these connections, we get

\[
\begin{align*}
\Gamma_{12} + \Gamma_{34} &= 2\ell_{s} dk^{s}, & -\Gamma_{12} + \Gamma_{34} &= 2n_{s} dm^{s}, \\
\Gamma_{43} &= -k_{s} dk^{s}, & \Gamma_{41} &= -m_{s} dm^{s}, \\
\Gamma_{31} &= -l_{s} dl^{s}, & \Gamma_{32} &= -n_{s} dn^{s}.
\end{align*}
\]

(5.23)

Now, the inverse tetrad \( \partial_{a} \) associated with our general flat null tetrad (5.13) is

\[
\partial_{a} = \frac{1}{\sqrt{2}} g^{\mu\lambda}(l_{A} m_{\lambda}, k_{A} n_{\lambda}, \ell_{\lambda} n_{\lambda}, -k_{A} m_{\lambda}) \partial_{\mu}.
\]

(5.24)

Now the result derived can be explicitly summarized as follows: with the four spinors restricted by the two normalization conditions

\[
a) \ k_{1} l_{2} - l_{1} k_{2} = 1, \quad b) \ m_{1} n_{2} - n_{1} m_{2} = 1,
\]

(5.25)

the forms \( e_{a}^{\mu} \) written in terms of null co-ordinates (5.2), are

\[
\begin{align*}
e^{1} &= k_{1} n_{1} d\zeta + k_{1} n_{1} d\xi - k_{2} n_{2} du + k_{1} n_{1} dv, \\
e^{2} &= \ell_{2} m_{1} d\zeta + \ell_{2} m_{1} d\xi - \ell_{3} m_{3} du + l_{1} m_{1} dv, \\
e^{3} &= -k_{2} m_{2} d\zeta - k_{1} m_{2} d\xi + k_{3} m_{3} du - k_{1} m_{1} dv, \\
e^{4} &= \ell_{2} n_{1} d\zeta + l_{1} n_{1} d\xi - \ell_{3} n_{3} du + l_{1} n_{1} dv.
\end{align*}
\]

(5.26)

The inverse tetrad is then

\[
\begin{align*}
\partial_{1} &= l_{2} m_{2} \partial_{\zeta} + l_{2} m_{2} \partial_{\xi} + l_{1} m_{1} \partial_{u} - l_{2} m_{2} \partial_{v}, \\
\partial_{2} &= k_{1} n_{2} \partial_{\zeta} + k_{3} n_{2} \partial_{\xi} + k_{1} n_{1} \partial_{u} - k_{2} n_{2} \partial_{v}, \\
\partial_{3} &= l_{1} n_{2} \partial_{\zeta} + l_{2} n_{1} \partial_{\xi} + l_{1} n_{1} \partial_{u} - l_{2} n_{2} \partial_{v}, \\
\partial_{4} &= -k_{1} m_{2} \partial_{\zeta} - k_{2} m_{2} \partial_{\xi} - k_{1} m_{1} \partial_{u} + k_{2} m_{2} \partial_{v}.
\end{align*}
\]

(5.27)
and the connections explicitly written are

\begin{align}
\Gamma_{42} &= - k_1 dk_1 + k_1 dk_2, \quad \Gamma_{31} = - l_2 dl_1 + l_1 dl_2, \\
\Gamma_{12} + \Gamma_{34} &= 2(- l_1 dk_2 + l_2 dk_1) = 2(- k_1 dl_2 + k_2 dl_1)
\end{align}

and

\begin{align}
\Gamma_{41} &= - m_2 dm_1 + m_1 dm_2, \quad \Gamma_{32} = - n_2 dn_1 + n_1 dn_2, \\
- \Gamma_{12} + \Gamma_{34} &= 2(- n_1 dm_2 + n_2 dm_1) = 2(- m_1 dn_2 + m_2 dn_1).
\end{align}

It can be observed that these formulae specialized to the case

\begin{align}
k_1 &= - i \gamma, \quad k_2 = i, \quad l_1 = i, \quad l_2 = 0, \\
m_1 = i \bar{\gamma}, \quad m_2 = - i, \quad n_1 = - i, \quad n_2 = 0
\end{align}

reduce to

\begin{align}
e^1 = d\zeta - \gamma dv, \quad e^2 = d\bar{\xi} - \bar{\gamma} dv, \\
e^3 = du + \bar{\gamma} d\zeta + \gamma d\bar{\xi} - \gamma\bar{\gamma} dv, \quad e^4 = dv
\end{align}

and

\begin{align}
\partial_1 &= \partial_\zeta - \gamma \partial_u, \quad \partial_2 = \partial_{\bar{\zeta}} - \bar{\gamma} \partial_u, \\
\partial_3 &= \partial_u, \quad \partial_4 = \partial_{\zeta} + \gamma \partial_{\bar{\zeta}} + \gamma \partial_u - \gamma\bar{\gamma} \partial_u,
\end{align}

with the connections

\begin{align}
\Gamma_{42} &= - d\gamma, \quad \Gamma_{32} = 0, \quad \Gamma_{41} = - d\gamma, \quad \Gamma_{31} = 0, \\
\Gamma_{12} + \Gamma_{34} &= 0, \quad - \Gamma_{12} + \Gamma_{34} = 0.
\end{align}

This particular tetrad, with $\xi$ understood as complex conjugate of $\zeta$ and $\bar{\gamma}$ as complex conjugate of $\gamma$, was used in (13) and is useful if one wants to exhibit one null tetrad vector in its general form, while representing the remaining members of the tetrad as simply as possible.

In the general case, instead of working with the spinors restricted by the normalization conditions, we can introduce six independent complex parameters $p$, $q$, $\psi$ and $\bar{p}$, $\bar{q}$, $\bar{\psi}$, in terms of which our spinors are given by

\begin{align}
\begin{vmatrix}
k_1 &= \frac{e \exp[\psi/2]}{\sqrt{p-q}}, \\
l_1 &= - \frac{e \exp[-\psi/2]}{\sqrt{p-q}}, \\
\rightarrow p &= - \frac{k_1}{k_2}, \quad q = - \frac{l_1}{l_2}, \quad \exp[\psi] = \frac{k_2}{l_2} \\
\end{vmatrix}
\end{align}

$\epsilon^2 = 1$. 

\begin{align}
\begin{vmatrix}
k_2 &= - \frac{e \exp[\psi/2]}{\sqrt{p-q}}, \\
l_2 &= \frac{e \exp[-\psi/2]}{\sqrt{p-q}}.
\end{vmatrix}
\end{align}
and

\[
\begin{align*}
\varepsilon_1 &= \frac{\varepsilon}{\sqrt{p - q}} \\
\varepsilon_2 &= \frac{-\varepsilon}{\sqrt{p - q}} \\
\varepsilon_3 &= \frac{-\varepsilon}{\sqrt{p - q}} \\
\varepsilon_4 &= \frac{\varepsilon}{\sqrt{p - q}}
\end{align*}
\]

(5.35)

\(\varepsilon^2 = 1\)

(The ambiguity of choice for \(\varepsilon = \pm 1\) in these formulae is related to the fact that the transformation \((k_X, l_X, m_X, n_X) \rightarrow (k_X, l_X, m_X, n_X)\) leaves \(\varepsilon^2\) invariant.)

In terms of the new parameters, \(\varepsilon\) become

\[
\begin{align*}
\varepsilon_1 &= \exp \left[ \frac{1}{2}(\varphi - \bar{\varphi}) \right] (du + \bar{q} \, d\zeta + p \, d\bar{\zeta} - p\bar{q} \, dv), \\
\varepsilon_2 &= \exp \left[ -\frac{1}{2}(\varphi - \bar{\varphi}) \right] (du + p \, d\zeta + q \, d\bar{\zeta} - q\bar{p} \, dv), \\
\varepsilon_3 &= \exp \left[ \frac{1}{2}(\varphi + \bar{\varphi}) \right] (du + \bar{q} \, d\zeta + p \, d\bar{\zeta} - p\bar{q} \, dv), \\
\varepsilon_4 &= -\exp \left[ -\frac{1}{2}(\varphi + \bar{\varphi}) \right] (du + \bar{q} \, d\zeta + q \, d\bar{\zeta} - q\bar{p} \, dv),
\end{align*}
\]

and the \(\partial\)'s are

\[
\begin{align*}
\partial_1 &= \exp \left[ -\frac{1}{2}(\varphi - \bar{\varphi}) \right] (\partial_1 + q \, \partial_2 + p \, \partial_3 - q\bar{p} \, \partial_4), \\
\partial_2 &= \exp \left[ \frac{1}{2}(\varphi + \bar{\varphi}) \right] (\partial_1 + p \, \partial_2 + q \, \partial_3 - q\bar{p} \, \partial_4), \\
\partial_3 &= -\exp \left[ -\frac{1}{2}(\varphi - \bar{\varphi}) \right] (\partial_1 + q \, \partial_2 + \bar{q} \, \partial_3 - q\bar{p} \, \partial_4), \\
\partial_4 &= \exp \left[ \frac{1}{2}(\varphi + \bar{\varphi}) \right] (\partial_1 + p \, \partial_2 + \bar{q} \, \partial_3 - p\bar{p} \, \partial_4).
\end{align*}
\]

(5.37)

With the tetrad parametrized in this manner, the connections are

\[
\begin{align*}
\Gamma_{45} &= \exp \left[ \varphi \right] \frac{dp}{p - q}, \\
\Gamma_{31} &= \exp \left[ -\varphi \right] \frac{dq}{p - q}, \\
\Gamma_{12} + \Gamma_{34} &= d\varphi + \frac{dp + dq}{p - q}.
\end{align*}
\]

(5.38)

and

\[
\begin{align*}
\Gamma_{31} &= \exp \left[ \varphi \right] \frac{dp}{\bar{p} - \bar{q}}, \\
\Gamma_{32} &= \exp \left[ -\varphi \right] \frac{dq}{\bar{p} - \bar{q}}, \\
-\Gamma_{12} + \Gamma_{34} &= d\bar{\varphi} + \frac{d\bar{p} + d\bar{q}}{\bar{p} - \bar{q}}.
\end{align*}
\]

(5.39)

Of course, formulae (5.36) up to (5.39) tacitly assume that, if we work with this parametrization of the flat (complex) tetrad, \(p, q\) and \(\bar{p}, \bar{q}\) are such that

\[
\begin{align*}
a) \quad p - q \neq 0, \\
b) \quad \bar{p} - \bar{q} \neq 0.
\end{align*}
\]

(5.40)
If $e^4$ and $e^5$ are selected as surface forming ($\Gamma_{124} = 0 = \Gamma_{423}$), then

\begin{equation}
(p_s = 0 = p_4) \rightarrow \left\{ \begin{array}{l}
(\partial_\xi - p \partial_u) p = 0 , \\
(\partial_\phi + p \partial_v) p = 0 .
\end{array} \right.
\end{equation}

(Demonstrating this implication, one uses (5.40).) By integrating these differential equations we conclude that $p = p(u, v, \zeta, \xi)$ must be determined by the (implicit) equation

\begin{equation}
F(p, u + p\xi, v - p^{-1}\zeta) = 0 ,
\end{equation}

with $F = F(a, b, c)$ being an arbitrary analytic function of the three variables.

6. – Conclusions.

The results of the last section present a convenient starting point for the study of the general DKS structures, i.e. the properties of the curvatures of these structures. At the present time, we are proceeding with further development of these ideas, being particularly interested in finding DKS structures which solve (complex) Einstein vacuum equations or (properly generalized to the complex case) electro-vacuum Einstein-Maxwell equations.

It may be pointed out that results of (6) already indicated the particular role of DKS structures in the study of « heavens ». There it was established that in a « strong heaven » one can always select the tetrad that so

\begin{equation}
\begin{align}
e^1 &= dp , \\
e^4 &= dq , \\
e^2 &= dx - \Theta_{uv} dp + \Theta_{xv} dq , \\
e^3 &= dy + \Theta_{uv} dp - \Theta_{ux} dq ,
\end{align}
\end{equation}

where \{pqxy\} are co-ordinates, and $\Theta = \Theta(pqxy)$ fulfills the second heavenly equation

\begin{equation}
\Theta_{xx} \Theta_{yy} - (\Theta_{xy})^2 + \Theta_{xp} + \Theta_{yx} = 0 .
\end{equation}

The curvature is then described by the (only nontrivial!) quantities

\begin{equation}
C^{(3)} = \Theta_{xxx}, \quad C^{(4)} = \Theta_{xxx}, \quad C^{(3)} = \Theta_{xyy}, \quad C^{(2)} = \Theta_{yyv}, \quad C^{(1)} = \Theta_{yvy} .
\end{equation}

The metric of « strong heaven », written in terms of the second key function

\begin{equation}
H : \quad ds^2 = 2 dp(dx - \Theta_{uv} dp + \Theta_{xv} dq) + 2 dq(dy + \Theta_{uv} dp - \Theta_{ux} dq) ,
\end{equation}
possesses *manifestly* the form of the DKS structure with the distinguished vectors being surface-forming gradients.

Therefore, the hope that in the complex extensions of general relativity the DKS structure may play a pertinent role appears to be slightly amplified by the argument presented above.

**RIASSUNTO (*)**

Si delineano i motivi basilari ed i fatti fondamentali della teoria delle metriche doppie di KS. Le metriche complesse \( n_{\mu} + 2Pk_{\mu}k_{r} + 2Ql_{\mu}l_{r} \) con \( k_{\mu} \) e \( l_{r} \) nulli e reciprocamente ortogonali diventano particolarmente interessanti quando \( k_{\mu} \) e \( l_{r} \) sono formatrici della superficie, cioè quando determinano una congruenza di corde nulle composte.

(*) Traduzione a cura della Redazione.

Резюме не получено.