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The formal properties of n-dimensional Wronskians and their geometric interpretation enable one to construct explicit analytic solutions to some nonlinear partial differential equations (PDE's) that generalize the Liouville equation \( \phi_{xx} = e^{2\phi} \). The studied PDE's are (a) \( \mathcal{L}_m^1 F = \text{const} \), (b) \( \mathcal{L}_m^1 F = G, \mathcal{L}_m^1 G = F \), and (c) \( \mathcal{L}_m^1 (\mathcal{L}_m^1 F) = \text{const} \), where \( \mathcal{L}_m^1 \) is the nonlinear differential operation \( \mathcal{L}_m^1 F := \det(\partial_i^1 \partial_j^1 F) \), with \( k, l = 0, 1, ..., m \). Some nontrivial formal properties of the composition of the \( \mathcal{L}_m^1 \)'s are established.

I. INTRODUCTION

The classical nonlinear Liouville partial differential equation (PDE)

\[
\phi_{xy} = e^{-2\phi} \quad \text{and} \quad \epsilon^2 = 1,
\]

(1.1)

which can be stated in the equivalent form

\[
FF_yy - F_x F_y = \epsilon, \quad \text{where} \quad F = e^\phi,
\]

(1.2)

is presently considered as one of the simplest examples of a PDE yielding solutions via the Bäcklund procedure.\(^2\)

It has been well-known for about 100 years that the PDE \([(1.1) \text{ and } (1.2)] \) has the remarkable property of possessing the solution

\[
F = (e^\phi)^{-1/2}(1 + pq) \Leftrightarrow \phi = \epsilon \ln[(e^\phi)^{-1/2}(1 + pq)],
\]

(1.3)

where the single variable functions \( p = p(x) \) and \( q = q(y) \) are arbitrary, constrained only by the condition \( e^\phi > 0 \), with the dots denoting the derivatives with respect to the corresponding variables. Thus the most general solution to the Liouville equation is algebraically constructed from arbitrary single variable functions and their derivatives.

Of particular interest are (i) the mechanism that assures us that the general solutions to (1.1) and (1.2) have the form of (1.3), and (ii) the existence of other PDE's with the general solutions of a similar structure, i.e., algebraically constructed from arbitrary functions of a single variable and their derivatives. This paper intends to offer at least a partial answer to these questions.

In earlier work with J. D. Finley on the problem of twisting N-type solutions in complexified general relativity, we encountered as an intermediate step the PDE's

\[
FF_yy - F_x F_y = G \quad \text{and} \quad GG_{xy} - G_x G_y = F.
\]

(1.4)

The structural similarity of these PDE's with (1.2) suggests that they be labeled as "double" Liouville equations. In fact, the general solution to these equations may be constructed in a fashion quite similar to that for the general solution to (1.2). More specifically, the solutions can be algebraically constructed from arbitrary single variable functions \( p_i(x) \), \( q_j(y) \), \( i = 1, 2 \), and their first and second derivatives.

The geometrical interpretation of the solutions to (1.4) has resulted in the elucidation of the formal properties of the three-dimensional Wronskians of functions of a single variable. It seems natural to develop a more general n-dimensional theory that contains PDE's (1.2) and (1.4) as special cases. Section II is a summary of the formal properties of the n-dimensional Wronskians and their basic minors, accompanied by the corresponding geometric interpretation. These are essential in Sec. III, which is concerned with the formal properties of the abstract nonlinear differential operators \( \mathcal{L}_m^1 \).

Of course, in terms of \( \mathcal{L}_m^1 \)'s, (1.2) and (1.4) may be stated as

(a) \( \mathcal{L}_1^1 F = \epsilon, \quad \epsilon^2 = 1 \) and (b) \( \mathcal{L}_1^1 F = G, \mathcal{L}_1^1 G = F \).

(1.5)

For \( m > 2 \), \( \mathcal{L}_m^1 \) constitutes the natural generalization of the concept of the "Liouville operator" \( \mathcal{L}_1^1 \). In Sec. IV we investigate the chains a-b-c of PDE's by exploiting these generalized Liouville operators. It is shown that we are able to determine the most general analytic form of their solution for the case of an a-chain. In the case of a b-chain, we are unable to determine some special solutions for \( m > 2 \), and the most general solutions for \( m = 1 \). Finally, we consider a c-chain for \( m = 1 \), which reduces to the biharmonic equation for the conformal factor of a two-dimensional Riemannian space. Also, we discuss alternative formulations of the differential problem under consideration. In Sec. V we discuss some open problems related to the results of this work. Semitrivial proofs of a computational nature are abbreviated by their basic ideas only. The nontrivial proofs are in appendices, which the interested reader might find useful in further work along the same lines as that given in the text.

The fact that the most general solution to (1.1) and (1.2) has the analytic form of (1.3) is quite useful in mathematical physics. In particular, within the theory of exact solutions in general relativity, experience has shown that whenever a Liouville equation occurs at an intermediate step, the corresponding problem is integrable up to the very end. In this respect, we believe that the techniques of this paper are to some extent manageable when applied to nonlinear Liouville-like PDE's and thus may find some useful applications.

\(^{On \text{ leave of absence from the University of Warsaw, Warsaw, Poland.}}\)
II. WRONSKIANS AND GEOMETRY

Let $F^n$ be the set of ordered $n$-tuples of the smooth functions of some single variable, say $x \in \mathbb{R}$. Thus a typical element $f \in F^n$ consists of $f = \{f_i(t), i = 1, \ldots, n\}$, where $f_i : \mathbb{R} \to \mathbb{R}$, and $n$ is a fixed integer. If the $f_i$'s are considered as meaningful modulo the arbitrary changes of the independent variable only, then, defining the equivalence class $C_f(t) = \{f_i(t(t'))\}$, where $t' : \mathbb{R} \to \mathbb{R}$ is an arbitrary smooth bijection, the set $F^n/C$ amounts to the set of smooth curves in $\mathbb{R}^n$.

The set $F^n/C$ automatically carries a rich structure induced by the concepts of the Wronskian and its basic minors. These concepts are understood as the mappings $W$:

(a) $W(\lambda f) = \lambda^n Wf$,  
(b) $Wf' = \left(\frac{dt}{dt'}\right)^n(t) Wf$,  
(c) $(\lambda f_i) = \lambda^{-1} f_i$,  
(d) $W(Mf) = \det M \cdot Wf$,  
(e) $W(\lambda^n f_i) = \lambda^{-1} f_i$,  
(f) $(f_i) = \left(\frac{dt}{dt'}\right)^n(t) f_i$,  
(g) $W(Mg f_i) = \det M \cdot M_g^{-1} f_i$,  
(h) $f_i = \left(\frac{-1}{n-1}\right)(Wf)^{n-1}$.

In the first line of Eqs. (2.2), a smooth $\lambda : \mathbb{R} \to \mathbb{R}$ is arbitrary and $\lambda f = \{\lambda f_i(t)\} \in F^n$. In the second line, given $f = \{f_i(t)\}$, a smooth bijection $t : \mathbb{R} \to \mathbb{R}$ induces $f' = \{f_i(t'(t))\} = \{f_i(t(t'))\}$ in the left-hand members $W$. The $\lambda$ is meant as the nonlinear differential operation with respect to $t'$, while in the right-hand members it refers to the variable $t$. Note that (2.2)(b) is valid for $n \geq 2$, while (2.2) holds for $n > 3$. In the third line $M_f = \text{const}$ is an arbitrary nonsingular $n \times n$ matrix, with $Mf = \{M_{ij} f_i(t)\}$.

The first three lines of (2.2) follow directly from the definitions of the mappings $W$ and $\lambda$. The proofs of the identities given in the fourth line of (2.2) are nontrivial. They may be outlined in the form of a sequence of lemmas; those of interest are given in Appendix A.

$W$-regular curves: According to (2.2)(b), a smooth curve in $R^n$, represented as $f = \{f_i(t)\} \in F^n$, has the characteristic $Wf \neq 0$, independent of the choice for its parametrization.

Observe that when $\xi = \text{odd}$, the $\text{sgn}(Wf)$ cannot be affected by the change of the parametrization. This gives rise to the classification of the curves $f \in F^n/C$ into the two basic classes

$W$-regular: $Wf \neq 0$, $W$-singular: $Wf = 0$.

The origin of this classification is the condition $Wf \neq 0$ for $f \in F^n$, which is known to constitute a necessary and sufficient condition for the linear independence of $n$ smooth $f_i$'s, i.e., $Wf \neq 0 \Leftrightarrow \{\lambda_i = \text{const}, \lambda_i f_i(t) = 0 \Rightarrow \lambda_i = 0\}$. Consequently, a $W$-regular curve cannot be contained in any $(n-1)$-hyperplanes through the distinguished origin of $\mathbb{R}^n$. Correspondingly, each $W$-singular curve is contained in some $(n-1)$-hyperplane through the distinguished origin of $\mathbb{R}^n$.

Normal parametrization: Given a $W$-regular curve represented by $\{f_i(t)\} \in F^n$, $Wf \neq 0$, we propose to define its normal parameter $x \in \mathbb{R}$ via

$$dx = \text{sgn}(Wf) \cdot |Wf|^{1/2} dt.$$  

Heuristically, this idea is somewhat analogous to the idea of using the Pythagorean length as the natural parameter of the Frenet formulas and the concept of the relativistic proper time.

Performing the quadrature in (2.4), the derived function $x = t(x)$, with $dx/dt \neq 0$, defines its inverse $t = t(x)$. Thus the curve may be considered as given in terms of its normal parameter as $f' = f'(x) = f_i(t(x))$. Then it follows from (2.4) that

$$x' = \text{odd} \Rightarrow Wf' = 1$$  

and

$$x' = \text{even} \Rightarrow Wf' = \text{sgn}(Wf).$$

The differentiations in the operation $W$ are with respect to the variable $x$.

According to (2.2)(d), $\lambda$ maps $W$-regular curves into $W$-regular curves. Given a curve represented as $f \in F^n$, $Wf \neq 0$, we refer to $\lambda f \in F^n$, $W(\lambda f) \neq 0$ as the dual curve. Considering the curve $f$ as represented in terms of its normal parameter, $f' = \{f'_i(x)\}$, according to (2.2)(d) and (2.2)(h), we have

$$Wf' = 1 \Rightarrow \begin{cases} n = \text{odd}: W(\lambda^2 f_i) = 1, & \lambda f'_i = -f'_i, \\ n = \text{even}: W(\lambda^2 f_i) = 1, & \lambda f'_i = -f'_i, \end{cases}$$

and

$$Wf' = -1 \Rightarrow \begin{cases} n = \text{odd}: W(\lambda^2 f_i) = 1, & \lambda f'_i = -f'_i, \\ n = \text{even}: W(\lambda^2 f_i) = -1, & \lambda f'_i = -f'_i. \end{cases}$$

Therefore, $\lambda$ is an involution or anti-involution among the $W$-regular curves and their duals. Note that according to
(2.5), the formulas (2.7) are of interest only when \( \tau \) is even \( \Rightarrow n = 4, 5, 8, 9, \ldots \).

The mapping: The construction of the normal parameter \( x \) for a \( W \)-regular curve via (2.4) involves a quadrature. There is, however, a simple process that enables us to construct the \( W \)-regular curve, as given in terms of its normal parameter, bypassing the necessity of any integrations.

Consider a \( W \)-regular curve represented as \( f = \{ f_i(t) \} \in \mathbb{R}^n, W_f \neq 0 \). Then the mapping \( \downarrow \), defined by
\[
\nabla f_i = |W_f|^{-1} f_i,
\]
(2.8)

obviously produces another \( W \)-regular curve. The formal properties of \( \downarrow \) may be summarized in the form of the following theorem.

Theorem 2: The following identities hold:
(a) \( W(\downarrow) = sgn(W_f) \), (c) \( |\nabla f_i| = |W_f|^{-1} f_i \),
(b) \( \lambda(\nabla f_i) = sgn(x) \),
(d) \( \nabla f_i = |W_f|^{-1} f_i \),
(2.9)

Identity (2.9) (a) follows from (2.2) (a). Similarly, (2.9) (b) with arbitrary \( \lambda: \mathbb{R} \rightarrow \mathbb{R}, \lambda(t) \neq 0 \) follows from (2.2) (a). Equation (2.9) (c) is a trivial consequence of the definition of \( \nabla \). Equation (2.9) (d) may be established by using (2.2) (c) and (2.2) (d).

According to (2.4) and (2.9) (a), the curve \( g = \{ g_i(x) \} \) possesses the normal parameter \( x \), where \( x = sgn(W_f) \cdot t \). If we choose the integration constant for \( x \) equal to 0. Consequently, the curve
\[
g' = \{ g_i'(x) \} = \{ \nabla f_i \}_{i=1} = sgn(W_f) \cdot x \},
\]
(2.10)

according to (2.5), satisfies
(\( \tau \) is odd \( \Rightarrow Wf = 1 \),
(\( \tau \) is even \( \Rightarrow Wg = sgn(W_f) \).
(2.11)

The \((n-1)\)-dimensional interpretation of \( \nabla \): According to (2.9) (c), the mapping \( \downarrow \) has the nature of a projective operation. This induces its \((n-1)\)-dimensional interpretation. Indeed, \( Wf \neq 0 \) \& \( f_i \neq 0 \), and in particular \( f_i \neq 0 \). Thus we can represent the \( f_i ' s \) as \( f_i = |f_i|/W_f, f_i = |f_i|/W_f \). Then \( h_i = (\bar{h}_i, e), a = 1, \ldots, n - 1, e = 1, \) is an \( n \)-dimensional concept, \( \bar{h} = (\bar{h}_i) \in \mathbb{R}^n \), while \( h = (\bar{h}_i) \in \mathbb{R}^{n-1} \) is considered as a \((n-1)\)-dimensional concept. By \( \bar{h} \in \mathbb{R}^{n-1} \), we mean \( \bar{h} = (\bar{h}_i(t), \) with the dot denoting the derivative. Then one obtains
\[
\bar{W}h = e(-1)^{n-1} \bar{W}h,
\]
(2.12)

where the Wronskians of the left and the right are \( n \)- and \((n-1)\)-dimensional constructs, respectively. This being the case,
\[
|Wf| = |f_i| |W_f| = e(-1)^{n-1} |f_i| |W_f| \quad \text{[using (2.2) (a)].}
\]
(2.13)

Hence, \( Wf \neq 0 \) \( \Rightarrow Wf = 0 \). The last \((n-1)\)-dimensional concept has a simple geometric interpretation. Having \( \bar{W}h = 0 \) implies that there are nontrivial \( \bar{h}_a \) such that \( \lambda \bar{h}_a(t) = \lambda \bar{h}_a(t) = const \).

It follows that \( he^{n-1} \), with \( \bar{W}h = 0 \), is a curve in \( \mathbb{R}^{n-1} \) prohibited to be contained in any \((n-2)\)-hyperplane in \( \mathbb{R}^{n-1} \), and not only those through the origin of \( \mathbb{R}^{n-1} \).

Using the definition of \( \nabla \) and (2.13), one easily sees that
\[
\nabla f = |\bar{W}h|^{-1/n} h_a, \quad \nabla f = e|Wf|^{-1/n}.
\]
(2.14)

Therefore the functions (2.10), which automatically fulfill (2.11), may be considered as algebraically constructed from \( h_a(t) = \partial^2_{\lambda_a} (t), a = 1, \ldots, n - 1, k = 0, \ldots, n - 1 \), where the smooth \( h_a(t) \)'s are arbitrary, constrained only by the condition that the curve \( he^{n-1}/C \) is prohibited to be contained in any \((n-2)\)-hyperplane in \( \mathbb{R}^{n-1} \), equivalently \( Wh = 0 \). Indeed, one can also show that any \( W \)-regular curve in its normal parametrization, \( g' = \{ g_i'(x) \} \in \mathbb{R}^n \), so that (2.11) applies, there is \( h = \{ h_i(t) \} \in \mathbb{R}^{n-1} \), which "injects" it according to (2.10) and (2.14).

III. THE BASIC PROPERTIES OF \( \mathcal{L}_m \) OPERATORS

Let \( \mathcal{F} \) be the set of smooth functions of the two variables \( x \) and \( y \). Consider then a sequence of nonlinear differential mappings \( \mathcal{L}_m: \mathcal{F} \rightarrow \mathcal{F}, m = 1, 2, \ldots \), defined by
\[
F \in \mathcal{F} \Rightarrow \mathcal{L}_m f = \{ \det(\partial^m \partial^{n-m} F), k, l = 0, 1, \ldots, m > 1, \]
(3.1)

where \( \partial_m \) and \( \partial^n \) denote iterations of the differential operators \( \partial_x \) and \( \partial_y \). Of course, \( \partial_m \partial^n = \partial^m \).

It is convenient to extend the above definition of \( \mathcal{L}_m \)'s to all integer \( m \)'s, postulating that
\[
m = 0 \Rightarrow \mathcal{L}_m F = F, \quad m = -1 \Rightarrow \mathcal{L}_m F = 1, \quad m > -2 \Rightarrow \mathcal{L}_m F = 0, \]
(3.2)

Here, we will outline the basic formal properties of \( \mathcal{L}_m \)'s. One can easily see that the definition of \( \mathcal{L}_m \)'s implies the "homogeneity property":
\[
F_A(x), B(y) \in \mathcal{F} \Rightarrow \mathcal{L}_m AB = (AB)^{m+1} \mathcal{L}_m F, \quad m > -1, \]
(3.3)

if \( AB \neq 0 \), apply for \( m < -2 \).

Then one can show that under the change of the independent variables \( x = x(x'), y = y(y') \), \( \mathcal{L}_m \neq 0 \), meaning by \( \mathcal{L}_m, \mathcal{L}_m F = \det(\partial_m \partial^{n-m} F), m > 1, \) while (3.2) is valid with \( \mathcal{L}_m - \mathcal{L}_m \), the following identity holds:
\[
F \in \mathcal{F} \Rightarrow \mathcal{L}_m F = (\delta^{m-1}) \mathcal{L}_m F. \]
(3.4)

Next, one easily sees that \( \mathcal{L}_1 \) has the "distributive" property:
\[
F, G, F, G \in \mathcal{F} \Rightarrow \mathcal{L}_1 FG = F^2 \partial_x \partial_y \ln F. \]
(3.5)

This follows directly from \( \mathcal{F} \ni F \neq 0 \Rightarrow \mathcal{L}_1 F = F^2 \partial_x \partial_y \ln F. \) Also, note that \( \alpha = \text{const} \Rightarrow \mathcal{L}_1 \alpha = F^{2(\alpha - 1)} \mathcal{L}_1 F. \)

As far as the composition of the \( \mathcal{L}_m \) mappings is concerned, we claim that the basic identity
\[
\mathcal{L}_1 (\mathcal{L}_m F) = \mathcal{L}_{m+1} F, \quad \mathcal{L}_m F = \mathcal{L}_{m-1} F \mathcal{L}_m F. \]
(3.6)

is valid for every integer \( m \). The nontrivial proof of (3.6) is outlined in Appendix B, where we also discuss the general problem of the composition \( \mathcal{L}_m (\mathcal{L}_n F) \).

It follows from (3.6) and (3.5) that
\[
\mathcal{L}_2 (\mathcal{L}_m F) = (\mathcal{L}_{m+1} F)^2 \mathcal{L}_{m+1} F, \quad \mathcal{L}_{m+1} F, \quad (3.7)
\]

Indeed, (3.6) for \( m = 1 \) reduces to \( \mathcal{L}_1 (\mathcal{L}_1 F) = \mathcal{L}_2 F. \) Operating on both sides of (3.6) with \( \mathcal{L}_1 \) and then using the result on the left and (3.5) on the right, we have
\[\mathcal{L}_n F \cdot \mathcal{L}_m F = (\mathcal{L}_n F \cdot \mathcal{L}_m F)^2 (\mathcal{L}_m F \cdot \mathcal{L}_n F) \]  
\[= (\mathcal{L}_{n-1} F)^2 \mathcal{L}_m F \cdot \mathcal{L}_m F + (\mathcal{L}_m F)^2 \mathcal{L}_{m+1} F \cdot \mathcal{L}_{m+1} F \] 
\[+ (\mathcal{L}_m F)^2 \mathcal{L}_{m-2} F \cdot \mathcal{L}_{m-2} F \]  
[using (3.6)].  

(3.8)

Canceling this by \(\mathcal{L}_n F\) (in general \(\mathcal{L}_m F \neq 0\)), we obtain (3.7). Therefore, via the continuity argument, (3.7) is true for every \(F\) and every integer \(m\).

Crucial for our purposes, we state the properties of the \(\mathcal{L}_m\)'s in the form of two theorems.

**Theorem 3**: For every \(f_i(x)\) and \(g_j(y)\) (smooth), \(i = 1, \ldots, n\) and \(m > 1\), the identities

\[\mathcal{L}_m \sum_{i=1}^n f_i g_i = \begin{cases} 
0, & \text{if } m > n + 1, \\
W^f W^g, & \text{if } n = m + 1, \\
\sum_{i=1}^n f_i \cdot g_i, & \text{if } n = m + 2,
\end{cases}\]  

(3.9)

hold, where \(W\) and \(\ast\) are the mappings defined in Sec. I.

Employing the summation convention over the indices \(k_i = 0, \ldots, m = l\), we have, from the definition (3.1),

\[\mathcal{L}_m F = \left[1/(m + 1)!\right] \delta_{i_1 \ldots i_m} \cdot e_{k_1 \ldots k_{m+1}} \cdot \partial_x^{k_1} \partial_y^{k_{m+1}} F,\]  

where the \(e\)'s are \((m + 1)\)-dimensional Levi-Civita symbols normalized by \(e_{0 \ldots 0} = 1\). Consequently, with \(F = f_i(x) g_j(y)\), we have

\[\mathcal{L}_m f_i g_i = \left[1/(m + 1)!\right] \delta_{i_1 \ldots i_m} \cdot e_{k_1 \ldots k_{m+1}} \cdot \partial_x^{k_1} \partial_y^{k_{m+1}} f_i \cdot g_j,\]  

where \(j = 1, \ldots, n\) and \(i = 1, \ldots, n\).

(3.10)

In the above, \(f_i^j = (d/dx)^j f_i\), \(g_j^i = (d/dy)^i g_j\), and \([\ldots]\) denotes the antisymmetrization symbol of a set of indices.

When the \(i\)'s have the range \(i = 1, \ldots, n\), \(m = 1\), the antisymmetrization of \(m + 1\) of the indices of the above type automatically leads to 0. Thus the first line of (3.9) is true.

On the other hand, if the \(i\)'s have the range \(i = 1, \ldots, n = m + 1\), then according to (2.1),

\[f_i^j \cdot f_i^m_{m+1} \cdots f_i^{n-1}_{n+1} = \left[1/(m + 1)!\right] \delta_{i_1 \ldots i_m} \cdot W^f.\]  

Similarly,

\[g_i^j \cdot g_i^{m+1} \cdots g_i^{n-1} = \left[1/(m + 1)!\right] \delta_{i_1 \ldots i_m} \cdot W^g.\]  

(3.11)

Therefore, making the contraction of two \(e\)'s over \(m + 1\) indices, the second line of (3.9) follows from (3.11).

Finally, if the range of \(i\)'s is \(i = 1, \ldots, n = m + 2\), (3.11) can be rewritten employing the concept of the generalized Kronecker \(\delta\)'s in the form

\[\mathcal{L}_m f_i g_i = \delta_{i_1 \ldots i_m+1} \cdots \delta_{i_n+1 \ldots n} \cdot f_i^j \cdot f_i^m_{m+1} \cdots f_i^{n-1}_{n+1} \cdot g_j^i \cdot g_j^{m+1} \cdots g_j^{n-1}.\]  

(3.12)

On the other hand, the \(\delta\)'s are equivalent to the contraction of two \(e\)'s over one index. Hence

\[\mathcal{L}_m f_i g_i = e_{i_1 \ldots i_m+1} \cdots e_{i_1 \ldots i_m} \cdot f_i^j \cdot f_i^m_{m+1} \cdots f_i^{n-1}_{n+1} \cdot g_j^i \cdot g_j^{m+1} \cdots g_j^{n-1}.\]  

(3.13)

This, compared with the definition of * in (2.1), and remembering that presently \(n = m + 2\), assures the veracity of the first line of (3.9).

**Corollary 1**: Formula (3.12) remains valid for the range of \(i\)'s and \(j\)'s over \(1, \ldots, n\), and can be equivalently spelled out in the form

\[\mathcal{L}_m f_i g_i = \left[1/(n - m + 1)!\right] \times e_{i_1 \ldots i_n} \cdots e_{i_1 \ldots i_m+1} \cdot f_i^0 \cdot f_i^m_{m+1} \cdots f_i^{n-1}_{n+1} \cdot g_j^0 \cdot g_j^{m+1} \cdots g_j^{n-1}.\]  

(3.14)

For \(n > m + 2\), the objects \(e_{i_1 \ldots i_m} \cdots e_{i_1 \ldots i_n} \cdot f_i^0 \cdot f_i^m_{m+1} \cdots f_i^{n-1}_{n+1}\) are the generalized minors of the \(n \times n\) matrix \([f_i^j_{i_1 \ldots i_n}]_{i=1}^{n-1}, n\) within the objects of this paper, however, this generalization of the third line of (3.9) is of little importance.

**Corollary 2**: Iterating the third line of (3.9) and using (2.2) (h), it follows that

\[F = \sum_{i=1}^{n+2} f_i(x) g_i(y) \Rightarrow \mathcal{L}_m (\mathcal{L}_m F) = (W^f W^g)^m F.\]  

(3.15)

Therefore, \(\mathcal{L}_m\) is an involution among the functions of two-variables of the structure

\[F = \sum_{i=1}^{n+2} f_i(x) g_i(y),\]  

with \(f_i\)'s and \(g_i\)'s arbitrary, being constrained only by the condition \((W^f W^g)^m = 1\).

**Theorem 4**: For \(m > 1\),

\[\mathcal{L}_m F = 0 \quad \text{and} \quad \mathcal{L}_{m-1} F \neq 0 \Rightarrow F = \sum_{i=1}^m f_i(x) g_i(y),\]  

(3.16a)

\[\mathcal{L}_m F = \text{const} \neq 0 \Rightarrow F = \sum_{i=1}^{m+1} f_i(x) g_i(y),\]  

(3.16b)

the implications being understood in the sense of the existence of the corresponding functions of the one variable, constrained in the case of (3.16) (a) by the condition \(W^f W^g \neq 0\). Similarly, in the case of (3.16) (b) by \(W^f W^g = \text{const}\).

The proof of (3.16) (a) is given in Appendix C. Once the veracity of (3.16) (a) is granted, a simple proof of (3.16) (b) follows by employing the identity (3.6).

Indeed, with \(\mathcal{L}_m F = \text{const} \neq 0\), \(\mathcal{L}_m F = 0\) obviously requires \(\mathcal{L}_{m-1} F \cdot \mathcal{L}_{m+1} F = 0\). If we held to hold with \(\mathcal{L}_{m-1} F = 0\), then according to (3.16) (a) \(F\) would have the most general form of

\[F = \sum_{i=1}^{m-1} f_i(x) g_i(x),\]  

which then according to the first line of (3.9) leads to \(\mathcal{L}_m F = 0\), contradicting \(\mathcal{L}_m F = \text{const} \neq 0\). Therefore, \(\mathcal{L}_{m+1} F = 0\), so that according to (3.16) (a) \(F\) has the most general form

\[F = \sum_{i=1}^{m+1} f_i(x) g_i(y).\]  

But then according to the second line of (3.9) \(\mathcal{L}_m F = W^f W^g = \text{const}\).
We conclude this section recognizing the fact that if \(0 \neq F, G \in \mathbb{F}^m\) are related by the condition
\[
\mathcal{L}_F = \mathcal{L}_1 G, \tag{3.17}
\]
then, parametrizing equivalently these objects according to
\[
F = e^{-\varphi/2} \cosh(\phi/2), \quad G = e^{-\varphi/2} \sinh(\phi/2), \quad \phi \neq 0,
\tag{3.18}
\]
condition (3.17) can be stated in the simple form of
\[
\psi_{xy} = \phi_x \phi_y. \tag{3.19}
\]

IV. THE LIOUVILLE-LIKE PDE's

This section examines some PDE's constructed by using the notion of the nonlinear differential operators \(\mathcal{L}_m, m \geq 1\).

We define first as the basic chain of the Liouville-like PDE's for the searched \(F, G \in \mathbb{F}^m\):
\[
\mathcal{L}_m F = \epsilon_m, \quad \epsilon_m = 1, \quad m = 1, 2, \ldots, \tag{4.1}
\]
These equations together with the associated "degenerate" chain
\[
\mathcal{L}_m F = 0, \quad m = 1, 2, \ldots, \tag{4.2}
\]
include PDE's of the form
\[
\mathcal{L}_m F = \text{const,} \quad m = 1, 2, \ldots, \tag{4.3}
\]
If \(F\) fulfills (4.3) with \(\text{const} \neq 0\), then \(F' = aF, a = \text{const}, \) fulfills \(\mathcal{L}_m F' = \text{const} \cdot a^{m+1}.\) Therefore, choosing \(a\) properly and dropping the prime with \(\text{const} \neq 0\), (4.3) reduces to (4.1). Obviously, the proper Liouville equation (1.2) constitutes the first member of the chain (4.1) for \(m = 1\).

Now, we propose to consider a chain of PDE's for searched \(F, G \in \mathbb{F}^m\):
\[
\mathcal{L}_m F = G, \quad \mathcal{L}_m G = F, \quad m = 1, 2, \ldots, \tag{4.4}
\]
which generalize the "double" Liouville equations (1.4), equivalent to (1.5) (b).

It is also of some interest to comment from the point of view of this paper on the nature of the differential conditions for the searched \(F, G \in \mathbb{F}^m\):
\[
\mathcal{L}_m (\mathcal{L}_n F) = \text{const,} \quad m = 1, 2, \ldots. \tag{4.5}
\]
The first member of these PDE's for \(m = 1\), \(\mathcal{L}_1 (\mathcal{L}_2 F) = \text{const} \) will be seen to be equivalent to the bi-harmonic equation for the conformal factor of a two-dimensional Riemannian space, with the harmonic scalar curvature.

Of course, among the PDE's proposed above, the case of Eqs. (4.2) is the simplest. According to (3.16) (a), the most general solution to \(\mathcal{L}_m F = 0\) with \(\mathcal{L}_{m-1} F \neq 0\) has the form of
\[
F = \sum_{i=1}^m f_i(x)g_i(y), \quad \mathcal{L}_{m-1} F = Wf \cdot Wg \neq 0. \tag{4.6}
\]
In the terminology of Sec. II, it induces and is induced by the two \(W\)-regular curves in \(R^m, x \) and \(y \) playing the role of the arbitrary parameters of these curves prohibited to be contained in any \((m-1)\)-hyperplanes through the origin of \(R^m\).

Notice that given \(F\) in the form of (4.6), the functions
\[
f = \{f_i(x)\}, \quad g = \{g_i(y)\} \in \mathbb{F}^m
\]
are meaningful modulo the affine transformations only:
\[
f' = \mathcal{M}f, \quad g' = (M^{-1})^T g, \quad M = \text{const} \in GL(m), \tag{4.7}
\]
the matrices \(M\) being otherwise arbitrary.

Next, examining PDE's (4.1), we observe that according to (3.16) (b) the most general solution must have the form of
\[
F = \sum_{i=1}^{m+1} f_i(x)g_i(y), \tag{4.8}
\]
where \(f = \{f_i(x)\}, \quad g = \{g_i(y)\} \in \mathbb{F}^{m+1}\) (according to the second line of (3.9) ] are constrained by the condition
\[
Wf \cdot Wg = \epsilon, \tag{4.9}
\]
and are otherwise arbitrary.

Again, given the solution to (4.1) in the form (4.8) and (4.9), one easily sees that \(f\) and \(g\) are meaningful modulo the transformations (4.7) only, but this time with the \((m+1) \times (m+1)\) matrix \(M = \text{const} \in GL(m+1).\) Condition (4.9) obviously requires that both \(Wf\) and \(Wg\) be constants \(\neq 0.\) Using then as the special case of the transformations (4.7), \(f_i, -\lambda f_i, \lambda \cdot -g_i, \lambda = \text{const} \neq 0,\) one easily sees that, without losing any generality, we can always arrange that (4.9) constraining the general form of \(F\) from (4.8) is fulfilled with
\[
(Wf)^2 = 1 = (Wg)^2. \tag{4.10}
\]
But with the above being valid, according to the results of Sec. II, \(f, g \in \mathbb{F}^{m+1}\) may be interpreted as the two \(W\)-regular curves in \(R^{m+1}\) given, respectively, in terms of their normal parameters \(x\) and \(y.\) Thus we can interpret \(f\) and \(g\) from \(\mathbb{F}^{m+1}\) in (4.8) as the two arbitrary \(W\)-regular curves in \(R^{m+1},\) which are forbidden to be constrained in any \(m\)-hyperplanes through the origin of \(R^{m+1},\) as given in terms of their normal parameters. Notice that with this interpretation, the most general form of the solution to (4.1) as given by (4.8) constrained by (4.10), with \(f\) and \(g\) from \(\mathbb{F}^{m+1}\) in (4.8) as the two arbitrary \(W\)-regular curves in \(R^{m+1},\) which are forbidden to be constrained in any \(m\)-hyperplanes through the origin of \(R^{m+1},\) is constrained by \((\text{det } M)^2 = 1.\)

With the objectives outlined in the Introduction in mind, the basic point of this section is that, according to the properties of the mapping \(B\) given in Sec. II [i.e., that a \(W\)-regular curve in \(R^{m+1}\) (as given in terms of its normal parameter) can be always algebraically constructed from \(m\) smooth functions and their derivatives up to \(m+1\) order], the most general solution to (4.1) can be equivalently stated in the form
\[
F = (e Wp \cdot Wq)^{-1/(m+1)} \left(1 + \sum_{a=1}^{m+1} p_a(x)q_a(y)\right), \quad m > 1. \tag{4.11}
\]

The \(p = \{p_a(x)\}\) and \(q = \{q_a(y)\} \in \mathbb{F}^m\) in the above are two arbitrary smooth curves in \(R^m,\) forbidden to be constrained in any \((m-1)\)-hyperplanes such that \(e Wp \cdot Wq > 0.\) The solution is thus algebraically constructed from the arbitrary smooth functions \(p_a(x), q_a(y), a = 1, \ldots, m,\) and their derivatives \(\partial_a^k p, \partial_a^k q, k = 1, \ldots, m.\) This most general form of the solution to (4.1) for \(m = 1\) reduces precisely to (1.3), the classical result for the proper Liouville equation (1.2), \(ipo facto\) providing its proof, and hence

}\]
(4.11) is a natural generalization of (1.3) for the case of \( \mathcal{L}_m F = \epsilon \) PDE's.

**Corollary 1:** The result that the most general solution to (4.1) has the form (4.11), i.e., that \( F \) from (4.11) satisfies (4.1) and inversely, given \( F \) fulfilling (4.1), there are \( p, q \in \mathbb{F}^m \) such that (4.11) is true, is easily seen to admit a "complexification."

For simplicity, until now we have constrained in this text \( \mathbb{F}^n \) to be the set of real valued ordered \( n \)-tuples of smooth functions of the same real variable, and \( \mathcal{F} \) to be the set of real valued smooth functions of the real variables \( x \) and \( y \).

A moment of reflection, however, convinces us that if \( \mathbb{F}^n \) is interpreted as the set of the complex valued ordered \( n \)-tuples of the holomorphic functions of the same complex variable, and, correspondingly, \( \mathcal{F} \) is interpreted as the set of the complex valued holomorphic functions of the complex variables \( x \) and \( y \), then the most general solution to the "complexified" PDE (4.1), indeed has the form of (4.11), with the holomorphic functions \( \{ p_e(x) \}, \{ q_e(y) \} \in \mathbb{F}^m \) constrained by \( \epsilon WP \ Wq \neq 0 \), being otherwise arbitrary.

**Corollary 2:** The result that we are in possession of the most general solution to the PDE (4.1), either in its real or complexified version, has some interesting implications from the point of view of ODE's.

Suppose that we search the solution to (4.1) in the special case of \( F = F(x) \), \( z = x + y \), then (4.1) reduces to the nonlinear ODE of 2m differential order,

\[
\left| F, \frac{d}{dz} F, \ldots \frac{(dF)}{dz^m}, \frac{d}{dz} F, \ldots \frac{(dF)}{dz^m}, \frac{d}{dz} F, \ldots \frac{(dF)}{dz^m} \right| = \epsilon, \quad r^2 = 1.
\]

(4.12)

Our general result on the level of PDE (4.1) permits us to construct easily the explicit general solution to the ODE (4.12) as endowed with the 2m integration constants.

Indeed, if \( F \) from (4.8) depends only on \( z = x + y \), clearly \( f_i(x) \) and \( g_i(y) \) must have the form of

\[
f_i = \sqrt{p_i e^{a_i x}}, \quad g_i = \sqrt{p_i e^{a_i y}}, \quad F = \sum_{i=1}^{m+1} \beta_i e^{a_i x},
\]

(4.13)

where \( \alpha, \beta, i = 1, \ldots, m+1 \), are constants. With \( f_i \)'s and \( g_i \)'s of this form, one easily sees that

\[
Wf = M(\beta_1 \cdots \beta_{m+1})^{1/2} e^{(a_1 + \cdots + a_{m+1})x}, \quad Wg = M(\beta_1 \cdots \beta_{m+1})^{1/2} e^{(a_1 + \cdots + a_{m+1})y},
\]

(4.14)

where

\[
M := \begin{bmatrix} 1 & \cdots & 1 \\ \alpha_1 & \cdots & \alpha_{m+1} \\ \cdots & \cdots & \cdots \\ \alpha_1^m & \cdots & \alpha_{m+1}^m \end{bmatrix}
\]

is the Van der Mond determinant, so that the (4.9) condition amounts to

\[
\beta_1 \cdots \beta_{m+1} M^2 e^{(a_1 + \cdots + a_{m+1})z} = \epsilon.
\]

(4.15)

This necessitates for \( 2(m+1) \) constants \( \alpha_i \) and \( \beta_i \) the two conditions

\[
\alpha_1 + \cdots + \alpha_{m+1} = 0, \quad \beta, \beta \cdots \beta_{m+1}, M^2 = \epsilon, \quad (4.17)
\]

the second of these requiring obviously \( M \neq 0 \), so that necessarily \( i \neq j \Rightarrow \alpha_i - \alpha_j \neq 0 \). Modulo conditions (4.17), \( F \) from (4.13) solves (4.12), contains \( 2(m+1) - 2 = 2m \) arbitrary constants, and hence is the most general solution of the nonlinear ODE (4.12) of the 2m's differential order.

Perhaps one could guess the general shape of the solution to (4.12) in the form of (4.13) with the constants constrained by (4.17) *prima facie*, but in establishing this result, our knowledge of the most general form of the solution to the PDE (4.1) was certainly useful.

Now the case of the PDE's (4.4) is much more involved than the case of PDE's of the form (4.3), where we have succeeded in establishing the most general form of their solutions as endowed with a geometric interpretation. Since we are interested only in the nontrivial solutions to these equations, \( F \neq 0 \neq G \) [because (4.4) necessitates \( F = 0 \Leftrightarrow G = 0 \)], (i) by eliminating \( G \), we arrive at the necessary condition

\[
\mathcal{L}_m (\mathcal{L}_m F) = F, \quad m > 1,
\]

(4.18)

and (ii) if we constrain additionally the searched \( F \) and \( G \) by \( \mathcal{L}_{m+2} F = 0 \Leftrightarrow \mathcal{L}_{m+2} G = 0 \), then Eqs. (4.4) admit a special solution of the form

\[
F = \sum_{i=1}^{m+2} f_i(x) g_i(y), \quad G = \sum_{i=1}^{m+2} \ast f_i \ast g_i,
\]

(4.19)

where \( f_i, g_i \in \mathbb{F}^{m+2} \) are constrained by

\[
(Wf)^2 = 1 = (Wg)^2,
\]

(4.20)

and are otherwise arbitrary. This statement applies for \( m > 1 \).

Indeed, (4.18) is a trivial consequence of (4.4). On the other hand, with \( F \) of the form of (4.19) treated as an anzatz, according to \( G = \mathcal{L}_m F \) and the third line of (3.9), \( G \) must have the form of 4. Then according to (3.15), \( F = \mathcal{L}_m G = \mathcal{L}_m (\mathcal{L}_m F) = (Wf Wg)^m \) is also fulfilled iff \( (Wf Wg)^m = 1 \). This is equivalent to \( Wf \ Wg = \epsilon, \ r^2 = 1 \), with \( m \) odd, \( \epsilon \) constrained to the value \( \epsilon = 1 \). Rescaling \( \lambda_i \rightarrow \lambda_i, g_i \rightarrow -g_i, \lambda = \text{const} \neq 0 \), we can always arrange that the last condition be fulfilled with (4.20) being valid.

The special solution to (4.4) described by (4.19) and (4.20) with \( Wf \ Wg = \epsilon, \ r^2 = 1 \), has of course a parallel interpretation to that given before to the solutions to \( \mathcal{L}_m F = \epsilon \), with \( m = m + 1 \). Thus \( F \) and \( G \) are induced by two \( W \)-regular curves in \( R^{m+2} \) prohibited to be contained in any \( (m+1) \)-hyperplane through the origin, with \( x \) and \( y \) serving as their normal parameters. From (4.19), we observe that \( F \) may also be interpreted as given in the form of (4.11) with \( m = m + 1 \). Of course, the corresponding \( G = \mathcal{L}_m F \) can then be evaluated in terms of \( \{ p_e(x) \}, \{ q_e(y) \} \in \mathbb{F}^m \). Observe also that \( F \) and \( G \) from (4.19), with \( f_i g_i \in \mathbb{F}^{m+2} \), have the relative symmetric structure, compatible with the symmetry \( F \rightarrow G, G \rightarrow F \) of Eqs. (4.2), due to the involutory relations (2.6) and (2.7), which apply because of (4.20).

The question arises, "How general is the solution (4.19) and (4.20) to (4.4)?" Answering this, we claim that, for \( m = 1 \), the solution constructed above is the most general solution to (4.4), which arose from a problem in general
relativity and motivated our interest in the chain of PDE's (4.4).

Indeed, in the case of \( m = 1 \), (4.4) reduces to (1.5) (b) and the identity (3.6) reduces to

\[
L_1(L_2 F) = F L_2 F.
\]  
(4.21)

Consequently, (4.18) with \( F \neq 0 \) is equivalent to

\[
L_2 F = 1,
\]  
(4.22)

which was shown to possess the most general solution

\[
F = \sum_{i=1}^{3} f_i(x) g_i(y),
\]

\[
WF = Wg = 1,
\]

equivalently, \( WF = \epsilon = Wg \), \( \epsilon^2 = 1 \). But with \( n = 3 \), according to (2.5), the normal parameters can be so selected that the \( \epsilon \) above is constrained to the value \( \epsilon = 1 \). Thus the most general solution to (1.5) (b) has the form of

\[
F = \sum_{i=1}^{3} f_i(x) \cdot g_i(y), \quad G = \sum_{i=1}^{3} *f_i *g_i,
\]

\[
WF = 1 = Wg.
\]

For the \( W \)-regular curves \( f_i g_i \in \mathbb{F}^3 \) given in terms of their normal parameters \( x \) and \( y \), and with the first line of (2.6) being in * involution to the dual curves *\( f_i *g_i \in \mathbb{F}^3 \), *\( *f = f \) and *\( **g = g \).

The above is a rather nice result. Our original problem from general relativity admits the most explicit general solution endowed with a simple geometric interpretation. The solution to (1.5) (b) induces—and is induced by—the two arbitrary \( W \)-regular smooth curves in \( \mathbb{R}^3 \) given in terms of their normal parametrizations; \( F \) and \( G \) are constructed from these and their * dual curves. According to (4.7), given \( F \) and \( G \), \( f \) and \( g \) from \( \mathbb{F}^3 \) are determined, remembering that \( WF = 1 = Wg \) and, because of (2.2) (c), are arbitrary modulo (4.7) transformations with \( 3 \times 3 \) constant = \( \text{MeSL}(3) \). The geometric interpretation given above is thus meaningful modulo SL(3) transformations of \( \mathbb{R}^3 \). Of course, \( F \) from (4.23) can be also represented in the form of

\[
F = \left( Wp W^q \right)^{-1/3} \left( 1 + \sum_{a=1}^{2} p_a(x) q_a(y) \right),
\]  
(4.24)

while \( G = L_1 F \) can be elaborated in terms of \( p, q \in \mathbb{F}^2 \), accompanied by the corresponding geometric interpretation.

However, the argument considered above for the case of Eqs. (4.4) with \( m = 1 \) does not work in the case of these equations with \( m > 2 \). Consider, e.g., the case of (4.4) with \( m = 2 \). Condition (4.18), employing identity (3.7) specialized for \( m = 2 \), reduces to

\[
L_2 (L_2 F) = (L_1 F)^2 \cdot L_2 F + (L_3 F)^2 F = F.
\]  
(4.25)

Thus, when \( L_{m+2} F = L_t F = 0 \), indeed \( F \neq 0 \) constrained by \( L_2 F = \pm 1 \) is a solution. However, there is no \( a \) priori reason why \( L_2 F \) should be equal to zero. Similarly, there is no \( a \) priori reason why \( L_{m+2} F \) has to be equal to zero for \( m > 2 \).

In summary, we have established the most general solution to (4.4) for \( m = 1 \), and a nontrivial solution for \( m > 2 \). In the last case, the form of the general solution remains an open question.

**Corollary:** With \( F = : e^x \), \( G = : e^y \), (1.5) (b) assume the equivalent form of

\[
\phi_{xy} = e^{e-2x}, \quad \psi_{xy} = e^{e-2x}.
\]  
(4.26)

On the other hand, for \( m = 2 \), (4.4) using the identity (4.21), are equivalent to

\[
L_1 (L_2 F) = FG = L_1 (L_1 G),
\]  
(4.27)

or, with \( F = : e^x \), \( L_2 F = : e^3 \) and \( G = : e^y \), \( L_1 G = : e^x \), they assume the equivalent form of

\[
\phi_{xy} = e^{e-2x}, \quad \psi_{xy} = e^{e-2x}, \quad \lambda_{xy} = e^{e-2x}.
\]  
(4.28)

Similarly, (4.4) arbitrary \( m > 1 \) can be equivalently stated as a set of differential conditions of the second order, with the nonlinear terms involving the notion of exponentials.

Concluding this section, we should like to explain why these PDE's are of some interest in mathematical physics. Given a two-dimensional Riemannian space of signature \((+, −)\) in its conformally flat local representation in a chart \( \{x, y\} \),

\[
\Lambda^1 \phi \Lambda^1 g = 2e^{-2} (x, y) dx \otimes dy,
\]

the condition that its scalar curvature \( R \) is harmonic, \( R^{\alpha \beta} = 0 \), is easily seen to be equivalent to the biharmonic equation for the conformal factor, \( \phi^{\alpha \beta} \phi_{\alpha \beta} = 0 \), amounting to

\[
e^{2x} \partial_2 \partial_2 (e^{2x} \partial_2 \partial_2 \phi) = 0.
\]  
(4.29)

The above PDE is equivalent to the statement that the searched \( \phi \) fulfills

\[
L_1 e^{2x} = e^{2x} \partial_2 \partial_2 e^{2x} = A(x) - B(y),
\]  
(4.30)

where \( A \) and \( B \) are arbitrary smooth functions of one variable, in the general case such that \( AB \neq 0 \).

The differential problem (4.30) had emerged as relevant in general relativity in 1962,⁷ and, as is well known, constitutes the key to the general nontwisting solutions of the Petrov type III of the empty space-time Einstein equations. In a somewhat different context one should also see Brans.⁸

Up to now, Eq. (4.30) resists all attempts to construct its most general analytic solution. A special solution to (4.30) of the form

\[
e^x = \sqrt{\frac{2}{3}} (A - B)^{3/2} / \sqrt{AB}
\]

is well-known.

From the point of view of this paper, we observe first that with \( AB \neq 0 \), introducing in (4.30) the new independent variables \( x' = A(x) \), \( y' = B(y) \) and defining \( F = \exp[\phi + \frac{1}{2} \ln AB] \) after dropping out primes, the investigated PDE assumes the form of

\[
L_1 F = x - y.
\]  
(4.32)

It follows that \( L'_1 (L_1 F) = 1 \), which coincides with (4.5) for \( m = 1 \). This motivates our interest in the PDE's from the chain (4.5).

Of course, using identity (4.21), (4.32) implies

\[
L'_2 F = 1.
\]  
(4.33)

Notice that if we define \( F = : \sqrt{\frac{2}{3}} F^{3/2} \), then (4.32) assumes the form

\[
F' L_1 F' = x - y.
\]  
(4.34)
Acting on it with $\mathcal{L}_1$ and using (3.5) and (4.21), we infer the necessity of
\[ F^3\mathcal{L}_2 F' + (\mathcal{L}_2 F')^3 = 1. \tag{4.35} \]
This elucidates why $F': x - y \Rightarrow \mathcal{L}_1 F' = 1$ is a special solution, i.e., the mechanism of the solution (4.31), as stated in a slightly more general form.

On the other hand, if, instead of committing the independent variables to $x = A, y = B$, we just execute in (4.30) the transformation $x = x(x'), y = y(y'), \xi y' \neq 0$, one easily sees that by dropping out primes and with $F = e^\xi$ (4.30) assumes the form
\[ \mathcal{L}_1 F = \sum_{i=1}^n \kappa_i(x) \ell_i(y), \tag{4.36} \]
while the condition $A \neq 0$ is now equivalent to $W_k \cdot W_g \neq 0$.

It easily follows that the differential problem studied in its most general form and in coordinates arbitrary modulo $x = x(x'), y = y(y'), \xi y' \neq 0$ is equivalent to the conditions
\[ \mathcal{L}_1 F \neq 0 \Rightarrow \mathcal{L}_2 F \neq 0 \Rightarrow F \neq 0, \tag{4.37} \]
\[ \mathcal{L}_1^2(\mathcal{L}_1 F) = F^2 \mathcal{L}_3 F + (\mathcal{L}_2 F)^2 = 0. \tag{4.38} \]
Notice that because of (4.21), with $\mathcal{L}_1 F \neq 0$, $\mathcal{L}_2(\mathcal{L}_1 F) = 0$ is also equivalent to $\mathcal{L}_1(\mathcal{L}_1^2 F) = 0$.  

\section{V. CONCLUDING REMARKS}

The PDE's studied in this paper constructed with the help of $\mathcal{L}_m$ nonlinear differential operators are certainly of interest as they generalize in a natural manner the Liouville equation.

Using the properties of Wronskians, we were able to find the general solutions to some of these PDE's, i.e., Eqs. (4.3) and (4.4). The latter one especially is of great importance because of its role in the problem of type N spaces. We hope that further analysis of the problems presented here allows one to find solutions to much more involved cases.

The Liouville equation has been revealed as the important one in the study of the Born-Infeld massless scalar field and in the theory of relativistic strings.\footnote{Formulas (A1)} Note also that this equation in three and more dimensions is of interest in connection with the soliton and field theories.\footnote{Note also that this equation in three and more dimensions is of interest in connection with the soliton and field theories.}

We hope that our generalizations of the Liouville equation will find application not only in general relativity but also in many other domains of mathematical physics.

\section{APPENDIX A: PROOFS OF (2.2)(d) AND (2.2)(h)

\subsection*{IDENTITIES}

The basic difficulty in proving (2.2)(d) and (2.2)(h) for every $n \geq 2$ is due to the “proliferation” of $\epsilon$'s and the order of derivatives involved in the concepts of $\mathcal{W}(\ell \mathcal{S})$. More specifically, there is no obvious way to initiate the inductive process with respect to $n \geq 2$, and the usual combinatorics of $\epsilon$'s and related Kronecker generalized $\delta$'s cannot deal with the mentioned “proliferation” in an effective manner. Our proof will rely on some facts from the theory of linear ODE's, and the formal properties of the minors of the matrix of the Wronskian $||f_1||$, $i = 1, \ldots, n$, $k = 0, 1, \ldots, n - 1$, where $(f_i(t)) \in \mathbb{F}^n$. Within this proof, some “tangential” formal properties of the elements of $\mathbb{F}^n$, related to the mapping $\ast$, will emerge as of interest as such.

The \textit{minors} of the Wronskian are defined for every $(f_i(t)) \in \mathbb{F}^n$ by
\[ M_k^i = (-1)^k \epsilon_{\ell_1 \ldots \ell_{n-k}} f_{\ell_1} f_{\ell_2} \ldots f_{\ell_k} = 0, \quad i = 1, \ldots, n, \quad k = 0, 1, \ldots, n - 1, \tag{A1} \]
where the summation convention over $p$'s and $q$'s applies. (Furthermore, the symbols $M_k^{-1}$ and $M_k^i$, where the upper index exceeds the permitted range $k = 0, 1, \ldots, n - 1$, are to be understood as zero.) This definition assures us that
\[ f_i M_k^i = \delta_{ik} Wf, \tag{A2} \]
and the parallel
\[ \sum_{k=0}^{n-1} f_i M_k^k = \delta_{ij} Wf, \tag{A3} \]
with the obvious ranges for the free indices.\footnote{Observe that according to (2.1), $M_k^i = M_k^{-1} = s_i$. Note also that this equation in three and more dimensions is of interest in connection with the soliton and field theories.}

Observe that according to (2.1), $M_k^i = M_k^{-1} = s_i$.\footnote{Observe that according to (2.1), $M_k^i = M_k^{-1} = s_i$.}

[Formulas (A1) can be interpreted for $k = 0, 1, \ldots, n - 1$ as defining for $k = 0, 1, \ldots, n - 1$ the nonlinear differential mappings $\mathcal{W}_k^i = \mathbb{F}^n \rightarrow \mathbb{F}^n$. For our purposes the most important is the mapping $\mathcal{W}_{n-1} = \ast$, induced by the “basic minors” of the Wronskian.]

Observe also that with $|M_k^i|$ being the matrix of the minors of $f_k^i$ and $\det(f_k^i) = Wf$, an elementary identity holds, consistent with equalities (A2) and (A3).\footnote{Note also that this equation in three and more dimensions is of interest in connection with the soliton and field theories.}

After these comments concerned with the definitions and the basic properties of the minors of the Wronskian, we will now prove the following.

\subsection*{Lemma 1:}
Given any $f = \{f_i(t)\} \in \mathbb{F}^n$, there is $A = \{A_i(t)\} \in \mathbb{F}^n$, such that
\[ \sigma f_i = 0, \quad \sigma : = \partial_t^r - \sum_{j=1}^{r-1} A_j \partial_t^j, \tag{A6} \]
and, if $f_i$'s are linearly independent ($\ast \Rightarrow Wf \neq 0$), then
\[ A_n = \partial_t \ln(Wf), \tag{A7} \]

Suppose that $Wf \neq 0$. Then $h_i := |Wf|^{-1/n} f_i = \mathcal{W}f_i$, according to (2.2)(a), has the property $Wh = \text{const}.0$.

This differentiated $\partial_t$ amounts explicitly to
\[ \epsilon_{1 \ldots n} h_1 \ldots h_n = 1, \quad n \geq 2, \tag{A8} \]
where $h_i := (d/dt)^i h_i, k = 0, 1, \ldots, n$. From the properties of $\epsilon$ it follows then that the “vector” $h_i^k$ must be a linear combination of “vectors” $h_i^k, k = 0, 1, \ldots, n - 2$. Therefore, there is $A' = \{A'_i(t)\} \in \mathbb{F}^n$ such that
\[ h_i^k = - \sum_{j=1}^{n-1} A'_j h_j^k - 1 = 0, \tag{A9} \]

By substituting here $h_i = |Wf|^{-1/n} f_i$ and applying the Leibnitz rule for $\partial_t$ acting on a product of two functions, one easily verifies that (A9) reduces to (A6), with $A_n$, having the form of (A7).

Let $Wf = 0$, so that the $n$ of $f_i$'s are linearly independent. Also, let $f_i, i = 1, \ldots, n - 1$, be linearly independent, and hence there are $\lambda_1 = \text{const} \neq 0$ such that $f_n = \Sigma_{i=1}^{n-1} \lambda_i f_i$.  

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Repeating the argument given above, we infer the existence of $A^* = \{ f^*_j(t) \} \in \mathbb{F}^{n-1}$ such that, similar to (A6),

$$f_j^{*-1} - \sum_{j=1}^{n-1} A_j f_j^{*-1} = 0, \quad i = 1, \ldots, n - 1. \quad (A10)$$

But then $f_j = \sum_{i=1}^{n-1} \lambda_i f_i$ must a fortiori satisfy the same ODE (A10). Then acting on (A10) with $\partial_t$, we conclude that there is $A \in \mathbb{F}^{n-1}$ such that (A6) is valid. A trivial descending induction implies that it does not matter how many of $n f_i$'s are linearly independent. There is always $A \in \mathbb{F}^{n-1}$ such that (A6) is true for any $\{ f_i(t) \} \in \mathbb{F}^n$.

**Lemma 2:** We claim that, modulo the existence of $\{ f_i(t) \} \in \mathbb{F}^n$ established in Lemma 1, such that (A6) is valid for every $\{ f_i(t) \} \in \mathbb{F}^n$, it is true that

$$\partial_t A^k = - A^k \cdot A_{k+1} A^{k-1}, \quad (A11)$$

for all $k = 0, 1, \ldots, n - 1, \quad i = 1, \ldots, n$.

Indeed, differentiating $\partial_t$ using the definition (A1) and remembering the total skewness of $\epsilon$, we have

$$\partial_t A^k = - \epsilon \cdot \partial_t \cdot \partial_t A^k = - \epsilon \cdot \partial_t \cdot A^k \cdot A_{k+1} A^{k-1},$$

for every $k = 0, 1, \ldots, n - 1$, and $i = 1, \ldots, n$.

Therefore, if $M := \{ f_i(t) \}_{i=0}^{n-1}$, we have

$$\partial_t M^k = \partial_t M^k - A^k \cdot A_{k+1} M^{k-1}, \quad (A13)$$

and $A^k$ is a skew $k$-vector.

A trivial induction establishes then that this implies for every non-negative integer $l$:

$$L^l M^k = M^k - A^k \cdot A_{k+1} M^{k-1} - \cdots - A^k \cdot A_{k+1} M^{k-1} - A^k \cdot A_{k+1} M^{k-1} - A^k \cdot A_{k+1} M^{k-1}, \quad (A14)$$

Specializing this for $k = n - 1$, we have

$$L^l g_i = M^{n-1} - l + \left( \sum_{l=0}^{n-1} L^{l-1} A_{n-1} g_i \right). \quad (A15)$$

The last relation leads to the next lemma.

**Lemma 3:** There are the smooth functions $B^k_i(t)$ such that

$$g^*_i = - \epsilon \epsilon \epsilon \cdots \epsilon g_i. \quad (A16)$$

With (A15) valid, remembering $L := A_n - \partial_n$, (A16) is obviously true for $l = 0, \ldots, n - 1$. This established, we propose, as the last lemma needed, the following.

**Lemma 4:** It is true that

$$\epsilon \epsilon \epsilon \cdots \epsilon g_i = \left( \sum_{l=0}^{n-1} L^{l-1} A_l \right) g_i. \quad (A17)$$

Indeed, according to (A16), and remembering the total skewness of $\epsilon$, the last factor in the first line of (A17) can be replaced by $g_i^{n-2} \epsilon \epsilon \cdots \epsilon M^{n-1}$, the contributions from $B_i^{n-2}$ canceling out. By a parallel argument, proceeding from the right to the left, the second factor can be replaced by $g_i^{n-3} \epsilon \epsilon \cdots \epsilon M^{n-1}$, the contributions from $B_i^{n-3}$ canceling out. Proceeding inductively this way we end up with $g_i = - \epsilon \epsilon \epsilon \cdots \epsilon M^{n-1}$. Therefore, the right-hand member of the first line of (A17) amounts to

$$\epsilon \epsilon \epsilon \cdots \epsilon g_i = \left( \sum_{l=0}^{n-1} L^{l-1} A_l \right) g_i. \quad (A17)$$

Permuting now the factors from the last line to the opposite order, $M_1 \cdots M_{n-1}$ and remembering the total skewness of $\epsilon$, we conclude that the equality of $\epsilon \epsilon \epsilon \cdots \epsilon g_i$ to the second line of (A17) is true.

With the established veracity of Lemmas (1-4), the proof of identities (2.2) (d) and (2.2) (h) for any $n \geq 2$ is now very simple. Indeed, by contracting the equality of $\epsilon \epsilon \epsilon \cdots \epsilon g_i$ to the second line of (A17) with $M_i$, arbitrary, we have

$$M_i \epsilon \epsilon \epsilon \cdots \epsilon g_i = \left( \sum_{l=0}^{n-1} L^{l-1} A_l \right) g_i = \epsilon \epsilon \epsilon \cdots \epsilon g_i. \quad (A17)$$

Therefore, if $\omega \neq 0$, necessarily

$$\epsilon \epsilon \epsilon \cdots \epsilon g_i = \left( \sum_{l=0}^{n-1} L^{l-1} A_l \right) g_i = \epsilon \epsilon \epsilon \cdots \epsilon g_i. \quad (A17)$$

Via the continuity argument, this also must hold with $\omega \epsilon \epsilon \epsilon \cdots \epsilon M_1 \cdots M_{n-1}$ such that $\omega \epsilon \epsilon \epsilon \cdots \epsilon M_1 \cdots M_{n-1}$ is true for every $n \geq 2$. Similarly, we now easily prove (2.2) (d). Indeed, $\omega \epsilon \epsilon \epsilon \cdots \epsilon g_i = \epsilon \epsilon \epsilon \cdots \epsilon g_i$.

**Corollary:** Equality (A15) specialized for $l = n$ yields

$$L^n g_i = \left( \sum_{l=0}^{n-1} L^{l-1} A_l \right) g_i = \epsilon \epsilon \epsilon \cdots \epsilon g_i. \quad (A17)$$

Therefore, while to an arbitrary $\{ f_i(t) \} \in \mathbb{F}^n$ according to Lemma 1, there is associated a linear ODE
\( \phi_f = 0 \), \( \phi = \partial_i^n - \sum_{j=1}^n A_j \partial_i^{j-1} \) \hspace{1cm} (A23)

the functions \( g_i : = f_i \) must fulfill the somewhat conjugated linear ODE:

\[ \phi^* g_i = 0 \quad \phi^* = L^n - \sum_{j=1}^n L_j^{j-1} A_j, \] \hspace{1cm} (A24)

where \( L_i = A_n - \partial_i \). If \( Wf = \text{const} \Leftrightarrow A_n = 0 \), one can show that the operator \( \phi^* \) is just the adjoint of \( \phi \) in the standard sense, that, for every \( f, \geq F \),

\[ g \cdot \phi f - \phi^* g f = \partial_i h, \] \hspace{1cm} (A25)

where

\[ h = \sum_{k,l=0}^{n-1} h_{kl}(t) f^k(t) g^l(t), \]

\( h_{kl} \) independent of \( f \) and \( g \).

Perhaps the notion of \( \phi^* \) as “conjugated” to \( \phi \) may also be of interest when \( A_n \neq 0 \), but we shall not investigate this point in the present text.

**APPENDIX B: PROOF OF (3.6) IDENTITY**

Consider a matrix \( M_{ij} \), with the entries in a commutative field of numbers of characteristic zero, indexed by \( i, j = 1,\ldots,m \) as a positive integer. Understanding \( \epsilon_{i_1\ldots i_m} = \epsilon_{i_1\ldots i_m} \) as the totally skew \( m \)-dimensional Levi-Civita symbol normalized by \( \epsilon_{1\ldots m} = 1 \), and by \( \delta_{i_1\ldots i_k; j_1\ldots j_k} \) as the generalized Kronecker \( \delta \)'s, and assuming the summation convention, we have the basic identity

\[ \delta_{i_1\ldots i_k; j_1\ldots j_k} = 1/(m-k)! \epsilon_{i_1\ldots i_k} \epsilon_{j_1\ldots j_k; i_{k+1}\ldots i_m}, \]

\[ k = 0,1,\ldots,m. \] \hspace{1cm} (B1)

Moreover, the notion of the determinant of the matrix \( M_{ij} \) then has the role of a coefficient in the identities:

(a) \( \epsilon_{i_1\ldots i_m} M_{i_1 i_2}\ldots M_{i_m i_m} = \text{det}(M_{pq}) \epsilon_{i_1\ldots i_m}; \)

(b) \( \epsilon_{i_1\ldots i_m} M_{i_1 i_2}\ldots M_{i_m i_m} = \text{det}(M_{pq}) \epsilon_{i_1\ldots i_m}. \) \hspace{1cm} (B2)

The generalized minors of the matrix \( M_{ij} \) are then defined as

\[ m_{i_1\ldots i_k; j_1\ldots j_k} := 1/(m-k)! \epsilon_{i_1\ldots i_k} \epsilon_{j_1\ldots j_k; i_{k+1}\ldots i_m} M_{i_{k+1} i_{k+2}\ldots i_m}, \]

\[ k = 0,1,\ldots,m. \] \hspace{1cm} (B3)

Notice that for \( k = 0, m = \text{det}(M_{pq}) \), for \( k = 1, m_{i,j} \) are the minors of matrix in the conventional sense, and for \( k = m \), \( m_{i_1\ldots i_k; j_1\ldots j_k} = \delta_{i_1\ldots i_k; j_1\ldots j_k}. \)

Using this definition, one easily establishes with the help of (A1) and (A2) that

(a) \( M_{i_1 i_2}\ldots M_{i_k i_k} m_{i_k i_{k+1}\ldots i_m; j_1\ldots j_k} = \text{det}(M_{pq}) \cdot \delta_{i_1\ldots i_k; j_1\ldots j_k}, \)

(b) \( M_{i_1 i_2}\ldots M_{i_k i_k} m_{i_k i_{k+1}\ldots i_m; j_1\ldots j_k} = \text{det}(M_{pq}) \delta_{i_1\ldots i_k; j_1\ldots j_k}, \]

\[ k = 0,1,\ldots,m. \] \hspace{1cm} (B4)

These general rules imply that, in particular for \( k = 1, \)

\[ M_{ij} m_{ij} = \text{det}(M_{pq}) \cdot \delta_{ij} = M_{ij} m_{ij}, \] \hspace{1cm} (B5)

where \( \delta_{ij} \) are the standard Kronecker \( \delta \)'s. For \( k = 2, \) we have

\[ M_{x_i x_j} M_{y_i y_j} m_{x_i y_j; x_i y_y} = \text{det}(M_{pq}) \delta_{i_j i_y h_j}, \]

\[ = M_{x_i x_j} M_{y_j y_y} m_{x_i y_y; x_i y_j}. \] \hspace{1cm} (B6)

By contracting the first line of (B6) with \( m_{k, i}, m_{h, j} \) we obtain

\[ [\text{det}(M_{pq})]^2 m_{k, i, j, h} = \text{det}(M_{pq}) \cdot \delta_{i_j i_y h_j, m_{x_i y_y; x_i y_j}}. \] \hspace{1cm} (B7)

This, canceled by \( \text{det}(M_{pq}) \), in general \( \neq 0 \) via the continuity argument, and, remembering that

\[ \delta_{i_j i_y h_j} = \delta_{i_y i_y h_j} = \delta_{i_y h_j, \delta_{i_y h_j}}, \]

leads to the identity

\[ \text{det}(M_{pq}) \cdot m_{i_j i_y h_j} = m_{i_j i_y, i_y h_j, m_{i_j i_y, i_y h_j}}. \] \hspace{1cm} (B8)

which is essential for our purposes.

Now using the traditional notation of \( \cdot \cdot \cdot \) for the determinant of a matrix, one easily can show that the identity

\[ \begin{vmatrix} M_{ij} & A_i \\ B_j & 0 \end{vmatrix} = -m_{i, j} A_i B_j \] \hspace{1cm} (B9)

holds, with \( i \) enumerating the entries into the rows and \( j \) into the columns of the determinant of the \( (m + 1) \times (m + 1) \) matrix in the left-hand member of the identity above. The \( M_{ij}, A_i, \) and \( B_j \) are arbitrary with \( i, j = 1,\ldots,m \).

Slightly more difficult to demonstrate using the concepts above is the identity

\[ \begin{vmatrix} M_{ij} & A_i & C_i \\ B_j & 0 & 0 \\ D_j & 0 & 0 \end{vmatrix} = \begin{vmatrix} M_{ij} & A_i \\ B_j & 0 \\ D_j & 0 \end{vmatrix} \cdot \begin{vmatrix} M_{ij} & C_i \\ B_j & 0 \end{vmatrix} \] \hspace{1cm} (B10)

Again, the \( i \)'s enumerate the rows of the \( (m + 2) \times (m + 2) \) matrix on the left-hand side, whose determinant is to be taken.

Correspondingly, the \( j \)'s enumerate the columns. Of course, in (B10), \( M_{ij}, A_i, C_i, B_j, D_j \) are arbitrary with \( i, j = 1,\ldots,m \).

Contracting now (B8) with \( A_i, C_i, B_j, D_j \) and using (B9) and (B10), we arrive at the identity

\[ \begin{vmatrix} M_{ij} & A_i & C_i \\ B_j & 0 & 0 \\ D_j & 0 & 0 \end{vmatrix} = \begin{vmatrix} M_{ij} & A_i \\ B_j & 0 \\ D_j & 0 \end{vmatrix} \cdot \begin{vmatrix} M_{ij} & C_i \\ B_j & 0 \end{vmatrix} \] \hspace{1cm} (B11)

Now we claim, using the same notation, a more general identity

\[ \begin{vmatrix} M_{ij} & A_i & C_i \\ B_j & E & G \\ D_j & H & F \end{vmatrix} = \begin{vmatrix} M_{ij} & A_i \\ B_j & E \\ D_j & H \end{vmatrix} \cdot \begin{vmatrix} M_{ij} & C_i \\ B_j & F \end{vmatrix} \] \hspace{1cm} (B12)

with \( M_{ij}, A_i, C_i, B_j, D_j, E, F, G, \) and \( H \) arbitrary.

Indeed, (B12) is true because one easily sees via an elementary argument that the coefficients of the arbitrary \( E, F, G, \) and \( H \) are the same on both sides of (B12). It follows that for (B12) to hold it is sufficient to verify its validity with
\( E = F = G = H = 0 \). Of course, this amounts to the already established (B11).  

With the identity (B12), we can identify the entries of the \((m + 2) \times (m + 2)\) determinant in the left-hand member of (B12) with the entries of the determinant for \(\alpha_{m+1}F\) according to the scheme

\[
\begin{array}{c|c|c|c}
F \cdots \partial_{x}^{m-1}F & \partial_{x}^{m}F & \partial_{x}^{m+1}F \\
\cdots & \cdots & \cdots \\
\partial_{y}^{m-1}F \cdots \partial_{y}^{m-1}\partial_{x}^{m-1}F & \partial_{y}^{m}\partial_{x}^{m-1}F & \partial_{y}^{m+1}\partial_{x}^{m-1}F \\
\partial_{y}^{m}F \cdots \partial_{y}^{m} & \partial_{y}^{m}\partial_{x}^{m} & \partial_{y}^{m+1}\partial_{x}^{m} \\
\partial_{y}^{m+1}F \cdots \partial_{y}^{m+1}\partial_{x}^{m+1}F & \partial_{y}^{m+1}\partial_{x}^{m+1}F & \partial_{y}^{m+1}\partial_{x}^{m+1}F \\
\end{array}
\]

Using the definitions above, one easily sees that

\[
\mathcal{L}_{1}(\mathcal{L}_{m}F) = \mathcal{L}_{m}F \partial_{x} \mathcal{L}_{m}F - \partial_{x} \mathcal{L}_{m}F \partial_{x} \mathcal{L}_{m}F = \begin{vmatrix} M_{y} & A_{i} & C_{i} \\ B_{j} & E & G \\ D_{j} & H & F \end{vmatrix}. \tag{B13}
\]

By comparing (B13) and (B14) with the identity (B12), we conclude that for every \( F \in \mathcal{F} \) and \( m = 1,2, \ldots \) it is true that

\[
\mathcal{L}_{1}(\mathcal{L}_{m}F) = \mathcal{L}_{m-1}F \mathcal{L}_{m+1}F. \tag{B15}
\]

It is now convenient to formally extend the validity of this identity for all integers \( m \). Understanding \( \mathcal{L}_{0}F = F \), it is natural to define

\[
\mathcal{L}_{-1}F = 1, \quad \mathcal{L}_{-2}F = \mathcal{L}_{-3}F = \cdots = 0, \tag{B16}
\]

which consistently assures us that (B15) is indeed valid for all integer \( m \).

For \( m = 1 \), (B15) amounts to

\[
\mathcal{L}_{1}(\mathcal{L}_{1}F) = F \mathcal{L}_{2}F. \tag{B17}
\]

Substituting the above \( F \rightarrow \mathcal{L}_{m}F \), we have

\[
\mathcal{L}_{1}(\mathcal{L}_{1}(\mathcal{L}_{m}F)) = \mathcal{L}_{m}F \mathcal{L}_{2}(\mathcal{L}_{m}F). \tag{B18}
\]

On the other hand, using the representation of \( \mathcal{L}_{1}F \) in the form of \( F^{2} \mathcal{L}_{1} \mathcal{L}_{y} \) in \( F \), one easily finds that the operation \( \mathcal{L}_{1} \) has the property

\[
F, G \in \mathcal{F} \Rightarrow \mathcal{L}_{1}(FG) = G^{2} \mathcal{L}_{1}F + F^{2} \mathcal{L}_{1}G. \tag{B19}
\]

Consequently, acting on both sides of (B15) with \( \mathcal{L}_{1} \) and employing (B18) and (B19), we have

\[
\mathcal{L}_{m}F \mathcal{L}_{2}(\mathcal{L}_{m}F) = (\mathcal{L}_{m-1}F)^{2} \mathcal{L}_{1}(\mathcal{L}_{m+1}F) + (\mathcal{L}_{m+1}F)^{2} \mathcal{L}_{1}(\mathcal{L}_{m-1}F) = (\mathcal{L}_{m-1}F)^{2} \mathcal{L}_{m}F \mathcal{L}_{m+2}F + (\mathcal{L}_{m+1}F)^{2} \mathcal{L}_{m-2}F \mathcal{L}_{m}F \tag{B20}
\]

[using (B15)].

Canceling this by \( \mathcal{L}_{m}F \) in general \( \neq 0 \), via the continuity argument, we arrive at the identity

\[
\mathcal{L}_{2}(\mathcal{L}_{m}F) = (\mathcal{L}_{m-1}F)^{2} \mathcal{L}_{m+2}F + (\mathcal{L}_{m+1}F)^{2} \mathcal{L}_{m-2}F, \tag{B21}
\]

which is valid for all integers \( m \). In particular,

\[
\mathcal{L}_{2}(\mathcal{L}_{1}F) = F^{2} \mathcal{L}_{1}F + (\mathcal{L}_{2}F)^{2} \tag{B22}
\]

and

\[
\mathcal{L}_{2}(\mathcal{L}_{2}F) = (\mathcal{L}_{1}F)^{2} \mathcal{L}_{3}F + (\mathcal{L}_{2}F)^{2} \tag{B23}
\]

Notice that a direct proof of (B21) based only on the definition of \( \mathcal{L}_{m} \) operations, without employing (B15) and the (B19) property of \( \mathcal{L}_{y} \), would be highly nontrivial. In the initial stages of this paper, (B22) has been proved in particular via the direct computation, with the assistance of Dr. Alberto Garcia-Diaz, whose help is gratefully appreciated.

At this stage of trying to find some formal properties of \( \mathcal{L}_{k}(\mathcal{L}_{1}F) \), with at least one of the indices being an arbitrary integer, this question appeared to be an extremely messy algebraic problem.

The general problem of the result of the iteration of the \( \mathcal{L}_{m} \) operations, i.e., \( \mathcal{L}_{k}(\mathcal{L}_{1}F) = ? \), to which we have now the answer for \( k = 1,2 \) and \( l \) arbitrary, remains as a rather nontrivial algebraic problem.

One easily sees from the definition of \( \mathcal{L}_{m} \) that

\[
\mathcal{L}_{m}F = \mathcal{L}_{m-1}F \partial_{y} \mathcal{L}_{m}F + \cdots, \tag{B24}
\]

where "\( \cdots \)" denotes the terms constructed from the derivatives of \( F \) of the differential order \( < 2m - 1 \).

It follows that

\[
\mathcal{L}_{n}(\mathcal{L}_{m}F) = \mathcal{L}_{m-1}F \mathcal{L}_{n-1}(\mathcal{L}_{m}F) \partial_{y} \mathcal{L}_{m}F + \cdots, \tag{B25}
\]

where "\( \cdots \)" denotes the terms algebraically constructed from the derivatives of \( F \) of the differential order \( < 2(n + m) - 1 \). If we wish to express this statement in terms of the \( \mathcal{L}_{m} \) operators, multiplying (B25) by \( \mathcal{L}_{n-1}F \) and employing (B24), we arrive at

\[
\mathcal{L}_{n}(\mathcal{L}_{m}F) = \mathcal{L}_{m-1}F \mathcal{L}_{n-1}(\mathcal{L}_{m}F) \partial_{y} \mathcal{L}_{m}F + \cdots. \tag{B26}
\]

The dots denote the terms algebraically constructed from the derivatives of \( F \) of the differential order \( < 2(n + m) - 1 \).

On the basis of intuitive arguments, we conjecture that the "\( \cdots \)" terms described above consist of the algebraic constructs made of \( \mathcal{L}_{m}F, m = 0,1, \ldots, k < 2(m + n) - 1 \). We believe that it would be of interest to determine the explicit form of the "\( \cdots \)" terms in (B26), determining this way the "algebra" of the composition of the \( \mathcal{L}_{m} \) operators, \( (\mathcal{L}_{k} \circ \mathcal{L}_{m})F = \mathcal{L}_{n}(\mathcal{L}_{m}F) \), which is obviously associative.  

**APPENDIX C: PROOF OF (3.16a)**

Using the traditional explicit notation for the determinants for \( \mathcal{L}_{m}F \) as defined by (3.1), we have

\[
\mathcal{L}_{m}F = \begin{vmatrix} F, & \partial_{x}F, & \cdots, & \partial_{y}^{m}F \\ \partial_{y}F, & \partial_{x} \partial_{y}F, & \cdots, & \partial_{y} \partial_{y}^{m}F \\ \cdots, & \cdots, & \cdots, & \cdots \\ \partial_{y}^{m}F, & \partial_{y} \partial_{y}^{m}F, & \cdots, & \partial_{y}^{m} \partial_{y}^{m}F \end{vmatrix}. \tag{C1}
\]

Since the derivative of a determinant equals the sum of...
determinants with initial columns successively differentiated, we also have
\[
\partial_x \mathcal{L}_m F = \begin{bmatrix}
F, & \partial_x F, & \partial_x^{m-1} F, & \partial_x^{m+1} F \\
\partial_x F, & \partial_x \partial_x F, & \partial_x \partial_x^{m-1} F, & \partial_x \partial_x^{m+1} F \\
\partial_x^{m-1} F, & \partial_x \partial_x^{m-1} F, & \partial_x \partial_x^{m-1} F, & \partial_x \partial_x^{m+1} F \\
\partial_x^{m+1} F, & \partial_x \partial_x^{m+1} F, & \partial_x \partial_x^{m+1} F, & \partial_x \partial_x^{m+1} F
\end{bmatrix}
\]

Now \( \mathcal{L}_m F = 0 \) implies that the last column of (C1) must be a linear combination of the first \( m \) columns, i.e., there are \( A_j(x,y), j = 1, \ldots, m \), such that
\[
\mathcal{L}_m F = 0 \Rightarrow \partial_x^k \partial_x^{m+1} F = \sum_{j=1}^{m} A_j \partial_x^j \partial_x^{-1} F, \quad k = 0, \ldots, m.
\]
Similarly, if \( \partial_x \mathcal{L}_m F = 0 \), via the same argument there are \( A_j(x,y), j = 1, \ldots, m \), such that
\[
\partial_x^k \mathcal{L}_m F = 0 \Rightarrow \partial_x^k \partial_x^{m+1} F = \sum_{j=1}^{m} A_j \partial_x^j \partial_x^{-1} F, \quad k = 0, \ldots, m.
\]
In particular, \( (C2) \) for \( k = 0 \) is given by
\[
\mathcal{L}_m F = 0 \Rightarrow \partial_x^0 \mathcal{L}_m F = \sum_{j=1}^{m} A_j \partial_x^{-1} F.
\]
Acting on the above with \( \partial_x^k \) and applying Leibnitz's rule,
\[
\mathcal{L}_m F = 0 \Rightarrow \partial_x^k \mathcal{L}_m F = \sum_{j=1}^{m} \sum_{l=0}^{k} \binom{k}{l} \partial_x^{k-l} A_j \partial_x^l \partial_x^{-1} F,
\]
where the term in the right-hand side cancels with the left-hand member when the summation index \( l \) equals \( k \) because of (C2). Hence
\[
\mathcal{L}_m F = 0 \Rightarrow \sum_{j=1}^{m} \sum_{l=0}^{k} \binom{k}{l} \partial_x^{k-l} A_j \partial_x^l \partial_x^{-1} F = 0, \quad k = 1, \ldots, m.
\]
A parallel argument applies to the case of (C3). Specializing for \( k = 0 \), we have
\[
\partial_x \mathcal{L}_m F = 0 \Rightarrow \partial_x^0 \mathcal{L}_m F = \sum_{j=1}^{m} A_j \partial_x^{-1} F = 0, \quad k = 1, \ldots, m.
\]
and acting on it with \( \partial_x^k \),
\[
\partial_x^k \mathcal{L}_m F = 0 \Rightarrow \partial_x^k \partial_x^{m+1} F = \sum_{j=1}^{m} A_j \partial_x^j \partial_x^{-1} F
\]
Similarly, the entry in the summation over \( l, l = k \), because of (C3) cancels out with the left-hand member, and we are left with the conditions
\[
\partial_x \mathcal{L}_m F = 0 \Rightarrow \sum_{j=1}^{m} \sum_{l=0}^{k} \binom{k}{l} \partial_x^{k-l} A_j \partial_x^l \partial_x^{-1} F = 0, \quad k = 1, \ldots, m,
\]
which, with \( A_j = A_j, \) formally coincide with (C6).

**Lemma:** Conditions (C6) imply that, for every \( k = 1, \ldots, m \),
\[
h_{k,l} = \sum_{j=1}^{m} \partial_x^j \partial_x^{-1} A_j,
\]
where \( \partial_x^j \partial_x^{-1} F = 0 \), for every \( l = 0, \ldots, k - 1 \).

We prove the above by induction. For \( k = 1 \), (C6) reduces to
\[
\sum_{j=1}^{m} \partial_x A_j \partial_x^{-1} F = 0
\]
and (C10) is true, \( h_{1,0} = 0 \). Assume then (C10) for some \( k < k_0 \), \( 1 < k_0 < m \), i.e., \( h_{k,l} = 0, \quad l = 0, \ldots, k_0 - 1 \). However, according to the definition of \( h_{k,l} \), \( \partial_x \partial_x \partial_x \partial_x^{-1} F = h_{k,l} \).

It follows that
\[
h_{k+1, l} = (k+1)(k+1) - (k+1)(k+1) = (k+1)^2 - (k+1)^2 = (k+1)^2,
\]
we see that this condition implies \( h_{k+1,0} = 0 \). So that, according to (C12),
\[
h_{k+1, l} = 0, \quad l = 0, 1, \ldots, k_0
\]
which completes the inductive proof of (C10).

Now we are sufficiently prepared to demonstrate the veracity of (3.16)(a). Indeed, with \( \mathcal{L}_m F = 0 \), the condition (C6) according to (C10) implies
\[
\sum_{j=1}^{m} \partial_x A_j \partial_x^{-1} F = 0, \quad k = 1, \ldots, m.
\]
But the matrix \( ||\partial_x^k \partial_x^{-1} F||, k, l = 1, \ldots, m \), has determinant equal to \( \mathcal{L}_{m-1} F \). Therefore, if it is assumed that \( \mathcal{L}_{m-1} F \neq 0 \), the matrix is invertible. Then (C15) implies \( \partial_x A_j = 0 \Rightarrow A_j(x) = c_A \). It follows that \( F \) must satisfy (C4), which reduces to a linear ODE:
\[
\left( \partial_x^m - \sum_{j=1}^{m} A_j(x) \partial_x^{-1} F = 0
\right)
\]
Understanding by \( f_i(x), i = 1, \ldots, m \), the linearly independent solutions to this linear ODE, \( F \) must have the form of
\[
F = \sum_{i=1}^{m} f_i(x) g_i(y),
\]
where the \( g_i \)'s are "integration constants." Therefore (3.16)(a) is true.
3. "Automatically" in a parallel sense, e.g., a differential manifold automatically carries the Grassman-Cartan algebra.
4. A reader unaccustomed to work with Levi-Civita δ's and generalized Kronecker δ's may find the reference J. L. Synge and A. Schild, Tensor Calculus (Dover, New York, 1978) useful. Of course, in (2.1) the summation convention is assumed. Since we do not need to distinguish "covariant" and "contravariant" indices, the last convention is understood as applying with respect to the indices.
5. S. Hoene-Wroński was perhaps the first who was aware of this interpretation.
6. The condition \( \lambda f(t) = 0 \), with smooth \( f' \)'s and \( \lambda \) const, \( \Sigma_{t=0}^{\infty} |\lambda| \neq 0 \) implies \( \lambda f'(k) = 0, k = 0,1,\ldots,m-1 \). Hence \( \det(f') = \lambda f' \) must vanish. If \( \lambda f' = 0 \), the implication from the text follows.
14. The notion of \( f'^k_j = (d/dt)^k f_j \) forces \( k \) to have the range \( k = 0,1,\ldots,n-1 \) in (A2). The summation conversion applies with respect to indices written on "covariant" level. In (A3) we have used the explicit sum symbol dealing with "contravariant" indices. The validity of (A2) is a direct consequence of (A1). Equation (A2) is in the notation specialized for the considered case, with \( M_j^k \) being the minors of \( |f'^k_j| \), an elementary property of the minors of a matrix.
15. By taking the determinant of (A2), we have \( \lambda f' \det(M^k_j) = (\lambda f')^k \). Thus if \( \lambda f' \neq 0 \), \( \det(M^k_j) = (\lambda f')^{-k} \). By the continuity argument, this must be valid also with \( \lambda f' = 0 \).
16. Perhaps (B12) also may be of interest as an algorithm that facilitates the evaluation of \( (m+2) \times (m+2) \) determinants of rank \( m \), reducing it to the evaluation of the \( (m+1) \times (m+1) \) determinants.
17. A hint for the reader potentially interested in this point: the basic difficulty in generalizing \( \mathcal{L}_m F \) for \( n = 1,2 \), amounts to the fact that while \( \mathcal{L}_1 F \) has the "distributive" property (B19), it seems that \( \mathcal{L}_2 (F G) \) cannot be directly expressed as algebraically constructed from \( \mathcal{L}_m F, m = 0,1,2 \).