Symmetries of the nontwisting type N solutions with cosmological constant
Humberto Salazar I., Alberto García D., and Jerzy F. Plebaski

Citation: Journal of Mathematical Physics 24, 2191 (1983); doi: 10.1063/1.525930
View online: http://dx.doi.org/10.1063/1.525930
View Table of Contents: http://scitation.aip.org/content/aip/journal/jmp/24/8?ver=pdfcov
Published by the AIP Publishing

Articles you may be interested in
Gravitational waves in vacuum spacetimes with cosmological constant. II. Deviation of geodesics and interpretation of nontwisting type N solutions

Gravitational waves in vacuum spacetimes with cosmological constant. I. Classification and geometrical properties of nontwisting type N solutions

Electrovac type D solutions with cosmological constant

All nontwisting N's with cosmological constant

The nondiverging and nontwisting type D electrovac solutions with
Symmetries of the nontwisting type-Ν solutions with cosmological constant

Humberto Salazar I., Alberto García D.,* and Jerzy F. Plebański**
Centro de Investigación y de Estudios Avanzados del I.P.N., Departamento de Física. Apdo. Postal 14-740, 07000 México D.F., México

(Received 13 October 1982; accepted for publication 17 December 1982)

The conformal, homothetic, and isometric symmetries of all nontwisting type-Ν solutions are established. All solutions with \( \lambda \) allow at most the existence of two Killing vectors. All vacuum solutions, except the Robinson metric, permit maximally the existence of two (isometric and homothetic) symmetries. The Robinson solutions are the only ones which allow conformal symmetries.

PACS numbers: 04.20.Jb, 04.90. + e

1. INTRODUCTION

The aim of this paper is to describe systematically all possible symmetries of the nontwisting type-Ν solutions of the Einstein equations with \( \lambda \) (also allowing \( \lambda \) to be zero). This class of solutions, as was demonstrated in Ref. 1, subdivides into nine essentially different branches according to the scheme of contractions given below.

\[
\begin{align*}
NT(\lambda, Z, 1) & \rightarrow NT(0, Z, 1) \\
NT(\lambda, Z, 0) & \rightarrow NT(0, Z, 0) \\
NT(\lambda, Z, -1) & \rightarrow NT(0, Z, -1)
\end{align*}
\]

All these solutions have the property of being determined by an arbitrary complex function \( f \) which, in a chart of coordinates \( \{ X^\mu \} = \{ \xi, \bar{\xi}, r, t \} \), depends on \( \xi \) and \( t \) only, i.e., \( f = f(\xi, t) \).

The determination of the symmetries or motions of a given Riemannian space can be achieved by solving the Killing equations

\[
\nabla K g_{\mu\nu} = \kappa g_{\mu\nu} \quad (1.1)
\]

for some function \( \kappa = \kappa(X^\mu) \), where \( \nabla K \) denotes the Lie derivative with respect to the vector field \( K \). The \( g_{\mu\nu} \) are the components of the metric \( g \) which, in the null tetrad formalism with signature \(+ 2\), is given as

\[
g = 2e^1 \otimes e^1 + 2e^2 \otimes e^2 + e^3 \otimes \bar{e}^3, \quad e^4 = \bar{e}^4 \quad (1.2)
\]

where \( e^1, e^2, e^3, e^4 \) are the null tetrad vectors. The motion is conformal if the conformal factor \( \kappa \) is a function of the coordinates. The motion is homothetic or isometric according to whether \( \kappa \) is a nonzero constant or zero. Correspondingly, one says that a Riemannian space admits a conformal Killing vector (CKV), a homothetic Killing vector (HKV), or a Killing vector (KV).

For the class of metrics being studied, the problem of searching for symmetries reduces to solving a single constraint equation depending on the vector field \( K \) and the structural function \( f(\xi, t) \). For arbitrary general function \( f(\xi, t) \) all these solutions, except the \( R \) metric, have no symmetries. The existence of symmetries is related just to the constraint equation; every function \( f(\xi, t) \) satisfying the constraint equation allows at least one Killing direction.

In the next sections, the symmetries for each metric of the scheme of nontwisting solutions are established. The pattern we shall follow consists in giving the cotangent tetrad \( e^a \), the form-invariance metric transformations, the components of the Killing vectors, the conformal factor \( \kappa \), the constraint equation, the general solution to the constraint equation, and a table of results.

Complex functions will be designated by means of Greek symbols, while real functions will be labeled by Latin symbols; exception is made with respect to the structural function \( f(\xi, t) \) — complex, \( \psi(\xi, \bar{\xi}, r, t) \) — real, the real conformal factor \( \kappa \), the real cosmological constant \( \Lambda \), and the real function \( \Omega(t) \) appearing in the transformation of the variable \( t \). To designate constants we shall use the corresponding symbols with a suffix 0.

In the tables the symbols \( G, \kappa, \psi, \Lambda \), and \( \Omega \) stand for the maximal group order, the conformal factor, the conformal Killing vector, and the homothetic KV, respectively; the Killing vectors are given simply by their expressions. For the sake of simplicity, the suffix 0 of the constant is dropped; \( a, b, k, l, m, n \) are used to designate real constants, while \( \rho, \mu, \nu \) for complex ones. The structural functions quoted in the tables have been reduced to their simplest form by using the transformations of invariance of the metric; they are essentially different from one to another.

2. THE NT(\( \lambda \), Z, \( \epsilon \)) AND NT(0, Z, \( \epsilon \)) SOLUTIONS

The most general family of nontwisting \( N \) solutions, the \( \text{NT}(\lambda, Z, \epsilon) \) solutions, is given in a chart of coordinates \( \{ \xi, \bar{\xi}, r, t \} \) by the null tetrad

\[
\begin{align*}
e^1 &= \bar{e}^2 = r d\xi + (\psi_r - rt) dt, \\
e^2 &= \psi(\xi, \bar{\xi}, r, t) dt, \\
e^3 &= dr + \frac{1}{2} \left[ -\psi_{\bar{\xi}} + \frac{1}{2} \left( f_\xi + \bar{f}_\xi \right) + \frac{1}{4} \lambda r^2 \psi \right] dt
\end{align*}
\]

where \( f = f(\xi, t) \) is an arbitrary complex function depending on the variables \( \xi, t \).
on $\xi$ and $t$ only. Because the curvature is proportional to $f_{\xi \xi \xi}$, the space is a curved space if $f_{\xi \xi \xi}$ does not vanish. The function $\psi$ satisfying the Liouville equation, without any loss of generality, can be given in the form

$$\psi = 1 + \epsilon \xi \xi \xi,$$  \hspace{1cm} (2.2)

where the discrete parameter $\epsilon$ takes the values $1, 0, -1$, depending upon whether the source lines of the gravitational waves are, respectively, timelike, null, or spacelike. The quantities $\lambda$ and $Z$ appearing in the symbol $NT(\lambda, Z; e)$ stand for the cosmological constant and the divergence of the congruence $e^3$.

The nonvanishing component of the conformal curvature, by choosing a coordinate gauge such that $\psi$ is of the form (2.2), amounts to

$$C^{\xi\xi} = -(\rho)\xi \xi \xi f_{\xi \xi \xi}.$$  \hspace{1cm} (2.3)

The corresponding $NT(0, Z; e)$ vacuum metric is obtained from the $NT(\lambda, Z; e)$ solutions by equating $\lambda$ to zero.

The tetrad of these solutions, which generalize the Kundt metric and are restricted by the equation (2.6), are given by

$$t = t(t'), \hspace{1cm} \xi = \xi(\xi', t'), \hspace{1cm} \xi = \xi(\xi', t'), \hspace{1cm} r = (\xi^2 \xi) \frac{1}{2} r', \hspace{1cm} \psi = \xi t' (f' - t, \partial^2 \xi').$$  \hspace{1cm} (2.4)

For a metric with $\psi$ of the form (2.2), these transformations reduce to

$$t = t' + t, \hspace{1cm} \xi = \frac{\alpha(t)}{\xi \xi \xi}, \hspace{1cm} \xi' = \frac{\alpha(t) - \beta(t)}{\alpha(t) + \beta(t)}, \hspace{1cm} r = |\alpha + \beta|, \hspace{1cm} \psi = |\alpha + \beta|,$$  \hspace{1cm} (2.5)

$$f = (\alpha + \beta)^{-1} (f' - f_0), \hspace{1cm} f_0 = \alpha \beta - \alpha \beta + [\alpha - \alpha \beta + \epsilon \beta \beta - \beta \beta] \xi, \hspace{1cm} \xi = \xi(\xi', t').$$  \hspace{1cm} (2.6)

Searching for motions of the $NT(\lambda, Z; e)$ field, from the Killing equations one obtains the components of the $K$ vector

$$K^t = a(t) = \frac{\Omega(t)}{\xi \xi \xi}, \hspace{1cm} K^\xi = \beta(t) = \epsilon \alpha(t) \xi^2 + ib(t) \xi + a(t), \hspace{1cm} K^\xi = (K^t)^t, \hspace{1cm} K^\xi = -r^2 f(t) \xi,$$  \hspace{1cm} (2.7)

Thus this class of solutions permits isometries only. For the $NT(0, Z; e)$ metric, $\kappa$ is equal to a constant, therefore these solutions permit at most the existence of homothetic Killing vectors.

The components of the $K$ vector and the function $f(\xi, t)$ are restricted by the equation

$$af_t + \beta f_{\xi} + (a - \beta) f = \beta.$$

The general solution $f(\xi, t)$, which allows maximally two symmetries, is given by

$$f(\xi, t) = (1/a \Omega) [\xi - \xi(t)]^2 [\phi(\chi) + \psi(\chi, t)],$$  \hspace{1cm} (2.8)

where $\xi(t)$ is any particular solution of the Ricci equation

$$\xi = a^{-1} (a(t) \xi^2 + ib(t) \xi + a(t))$$

the dot over $\alpha$ and $b$ denotes the derivative with respect to $t$, while the dot over $\beta$ is merely a symbol. In terms of them, the structural functions present in (2.7) are

$$\Omega(t) = \exp \int (a)^{-1/2} \beta dt, \hspace{1cm} \chi(\xi, t) = (\xi - \xi(t))^{-1} \Omega + \Pi,$$  \hspace{1cm} (2.9)

$$\Pi(t) = \int (a)^{-1/2} \beta \Pi dt,$$

$$\Psi(\xi, t) = \int [e \alpha \Omega - \Pi \beta + \Pi \beta] dt$$

[For a equal to zero, the function $f(\xi, t)$ happens to be $f = \beta$, $\beta$, $\beta_\xi$.]

For functions $f(\xi, t)$ of the form (2.7), the metric has symmetries (at most two $K$ vectors). For $f(\xi, t)$ outside of the mentioned class, the metric has no symmetries.

Not entering into details, by using the coordinate freedom, one obtains a list of particularly interesting structural functions, which are shown in Table I together with the corresponding symmetries. As was stated before the $NT(\lambda, Z; e)$ solutions permit only the existence of Killing vectors for $f(\xi, t)$ of the form given by (2.7) with the function $a$ being a constant. Therefore, the isometries of $NT(\lambda, Z; e)$ are the same as those of the $NT(0, Z; e)$ solutions.

3. THE $K(\lambda)$ SOLUTIONS

The tetrad of these solutions, which generalize the Kundt metric and reduce to it when $\lambda$ tends to zero, can be given as

$$e^t = (\xi^2) = \frac{d\xi}{\cosh x} = -r dt,$$

$$e^\xi = \mu^{-1} \tanh x dt,$$

$$e^z = \frac{dr}{\cosh x} + [\partial_\xi + \partial_x - 2\mu \tanh x] (f + f''),$$

$$-\mu r^2 \tanh x dt,$$

where $x = : \mu(\xi + \xi), \mu_\xi = (\xi_\xi)^{1/2}.$

The nonvanishing component of the conformal curvature is

$$C^{\xi\xi} = -2\mu \cosh x \sinh x.$$

The transformations which maintain the metric invariant are

Salazar I., Garcia D., and Plebański 2192

TABLE I. Symmetries of the $NT(\lambda, Z, \varepsilon)$ and $NT(0, Z, \varepsilon)$ solutions.

<table>
<thead>
<tr>
<th>$f(\xi, t)$</th>
<th>$f_{\xi\xi} \neq 0$</th>
<th>$G$</th>
<th>Homothetic and isometric killing vectors $\kappa = \text{const}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\phi(\xi)$</td>
<td>1</td>
<td>$T$</td>
<td>0</td>
</tr>
<tr>
<td>$t^{-1}\phi(\xi)$</td>
<td>1</td>
<td>$\mathcal{H} = tL$</td>
<td>2I</td>
</tr>
<tr>
<td>$e^{\omega t \xi \cdot t}$</td>
<td>1</td>
<td>$T - R$</td>
<td>0</td>
</tr>
<tr>
<td>$t^{-1} + 2z \xi \cdot t$</td>
<td>1</td>
<td>$\mathcal{H} = 1(L - 2R)$</td>
<td>2I</td>
</tr>
<tr>
<td>$\xi \cdot t$</td>
<td>2</td>
<td>$T, \mathcal{H} = 1(L + n^{-1}R)$</td>
<td>2I</td>
</tr>
<tr>
<td>$A^{-1}A(t) \ln \xi$</td>
<td>1</td>
<td>$A(t)R$</td>
<td>0</td>
</tr>
<tr>
<td>$\xi \ln \xi$</td>
<td>2</td>
<td>$T, e^{\omega t} R$</td>
<td>0</td>
</tr>
<tr>
<td>$t^{-1} \xi \ln \xi$</td>
<td>2</td>
<td>$T, e^{\omega t} M(\pm)$</td>
<td>0</td>
</tr>
<tr>
<td>$e^{-c}, \varepsilon = 0$</td>
<td>2</td>
<td>$T, \mathcal{H} = 1(L + D)$</td>
<td>2I</td>
</tr>
<tr>
<td>$t^{-1} + e^{-c}$, $\varepsilon = 0$</td>
<td>1</td>
<td>$\mathcal{H} = 1(L + D)$</td>
<td>2I</td>
</tr>
<tr>
<td>$\mathcal{H} = 1(L + n^{-1}R)$</td>
<td>2I</td>
<td>$A(t)R$</td>
<td>0</td>
</tr>
<tr>
<td>$\xi \ln \xi$</td>
<td>2</td>
<td>$T, e^{\omega t} M(\pm)$</td>
<td>0</td>
</tr>
<tr>
<td>$t^{-1} \xi \ln \xi$</td>
<td>2</td>
<td>$T, e^{\omega t} M(\pm)$</td>
<td>0</td>
</tr>
</tbody>
</table>

\[ t = \int e^{-\omega t'} \, dt', \quad \xi = \xi(t) = \xi(t') + i Z_{\xi}(t') \]
\[ r = e^{i\omega t'} r - (1/2\mu) \omega_{\xi} \sinh x, \]
\[ f = e^{2\omega t'} \left[ \left( f' - (1/8\mu^2)(\omega_{\xi'} + i[\omega_{\xi}, \xi]) + f_0 \right) \right], \quad (3.3) \]

where
\[ f_0(\xi', t') = \alpha(t') e^{2i\xi'} + \alpha(t') e^{-2i\xi'} + IT(t'), \]

this function is such that $C_1(\xi, f_0) = 0$.

By integrating the Killing equations one arrives at the components of the $K$ vector in the form
\[ \boldsymbol{K} = a(t), \]
\[ \boldsymbol{K} = (\frac{\partial}{\partial t}) = ib_{\xi}, \quad (3.4) \]
\[ \boldsymbol{K} = - r\mathbf{a} + (\partial/2\mu) \sinh x. \]

The conformal factor $\kappa$ must be zero. Therefore, this family of solutions permits at most isometries.

The components of $K$ and the function $f(\xi, t)$ are restricted by the constraint equation
\[ (\alpha f)' + ib_{\mu a}^{-1}(\alpha f)_{\xi} = \bar{\alpha} \alpha /8\mu^2 + (\alpha + 2i\mu b_{\mu a}^{-1}) e^{2i\xi} \]
\[ + (\bar{\alpha} - 2\mu b_{\mu a}^{-1} \bar{\alpha}) e^{-2i\xi} + iT(t'), \quad (3.5) \]

The general solution of the equation above is given by
\[ f(\xi, t) = a^{-2} \left[ \phi(\xi - ib_{\xi}) \left( \frac{dt}{\alpha} \right) + \frac{1}{8\mu^2}(\alpha - 1\bar{\alpha}) + f_0(\xi, t) \right], \quad (3.6) \]

where
\[ f_0(\xi, t) = \alpha(t') e^{2i\xi'} + \alpha(t') e^{-2i\xi'} + i T(t'). \]

Only the metric structures with functions $f(\xi, t)$ of the form (3.6) allow motions.

By using the coordinate freedom, one can establish that the maximal order of the group of isometries is 2. Concrete results are presented in Table II.

TABLE II. Symmetries of the $K(\lambda, \xi)$ solutions.

<table>
<thead>
<tr>
<th>$f(\xi, t)$</th>
<th>$f_{\xi\xi} \neq 0$</th>
<th>$G$</th>
<th>Killing vectors $\kappa = 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\phi(\xi - ib t)$</td>
<td>1</td>
<td>$T + bR$</td>
<td>0</td>
</tr>
<tr>
<td>$A(t) \xi$</td>
<td>1</td>
<td>$R$</td>
<td>0</td>
</tr>
<tr>
<td>$\xi$</td>
<td>2</td>
<td>$T, R$</td>
<td>0</td>
</tr>
<tr>
<td>$e^{\omega t}$</td>
<td>2</td>
<td>$T, R + nL$</td>
<td>0</td>
</tr>
</tbody>
</table>

\[ T = \partial_t, R = \partial_{\xi}, L = t \partial_t - \partial_{\xi} \]
4. THE KUNDT SOLUTIONS

The tetrad 1-forms, which describe the Kundt solutions, can be given as
\[ e^1 = (\xi') = d\xi - r \, dt, \]
\[ e^2 = (\xi + \xi') dt, \]
\[ e^3 = dr + (f + \tilde{f}) \, dt. \]
(4.1)
The conformal curvature, referred to the tetrad (4.1), is given by
\[ C^{(1)} = -(2/\xi + \xi') f_{\xi}. \]
(4.2)
The metric invariant transformations are
\[ t = e^{-\omega(t)} \, dt', \quad \xi = \xi' + i\omega, \quad \xi = \xi' - i\omega, \]
\[ r = e^{\omega(t)} \, [r' - \frac{i}{2}(\omega \gamma - e^{-\omega} + 1)(\xi' + \xi)], \]
\[ f = e^{-\omega(t)} \, (1 + [\xi']/4)(2\omega + (\omega)^2 + e^{-2\omega} - 1) \]
\[ + (4.3) \]
\[ + (I^T(t')). \]
The components of the K vector amount to
\[ K^t = a(t), \]
\[ K^\xi = \beta(\xi) = (\kappa/2)\xi + ib, \]
\[ K^r = (\kappa/2 - \bar{a}) \bar{r} + \frac{i}{2}(\xi + \xi')(\bar{a} + \bar{a}). \]
(4.4)
The conformal factor \( \kappa \) is a constant. Hence, the Kundt solutions permit at most the existence of homothetic Killing vectors.

The components of the K vectors and the structural function \( f(\xi, t) \) ought to satisfy the equation
\[ af_t + \beta f_t + (2a - \kappa/2)(f + \xi') \bar{\beta} - \bar{a} = 0, \]
\[ = ia^{-1}(1 - (a^{-1}/2)K(T + 2bS(t))), \]
where \( S(t) := \frac{1}{2}(2\ddbar{a} - a^2 - \bar{a}^2) \). The general solution of this equation is given by
\[ a^2f = e^{i(\kappa/2)(a^{-1} - a)} \phi \left( \ln \beta - \frac{\kappa}{2} \int a^{-1} \, dt \right) - \xi S(t) + iT(t). \]
(4.5)

For the existence of isometries, i.e., when \( \kappa \) is equal to zero, the function \( f(\xi, t) \) happens to be
\[ a^2f(\xi, t) = \phi (\xi - i\bar{b}) \int a^{-1} \, dt) - \xi S(t) + iT(t). \]
(4.6)

After using the transformations (4.3), one arrives at the canonical forms of the function \( f(\xi, t) \) and the corresponding symmetries shown in Table III.

5. THE ROBINSON SOLUTIONS

This section is included in the present paper for the sake of completeness. Most of the results concerned with isometries of the R solutions can also be found in Ref. 4.

The tetrad of the R solutions, the most "degenerate" nontwisting N's, can be given as
\[ e^1 = (\xi') = d\xi, \quad e^2 = dt, \quad e^3 = dr + (f + \bar{f}) \, dt, \]
(5.1)
with conformal curvature
\[ C^{(1)} = f_{\xi}. \]
The transformations preserving the form of the metric are
\[ t = a(t) \, dt', \quad \xi = e^{-ib}(\xi' + \frac{\beta(t')}{\xi'}), \]
\[ r = a_o^{-1} \, [r' - R(t') - \xi' \frac{\bar{\beta}}{\xi} \Bar{\beta}], \]
\[ f = a_o^{-1} \, (f' + \xi \frac{\bar{\beta}}{\xi} - \frac{i}{2} R + \bar{\beta} + it'), \]
(5.2)
dots denote the derivative with respect to \( t' \).
The components of the K vector are
\[ K^t = a(t) := i(t + m), \quad a(t) := i(t + m), \]
\[ K^\xi = \beta(\xi) = (\kappa/2)\xi + ib, \]
\[ K^r = (\alpha + \bar{a} - \bar{a}) \bar{r} - \frac{i}{2}(\xi + \xi')(\alpha + \alpha), \]
(5.3)

The conformal factor amounts to
\[ \kappa = \alpha + \bar{a} = 2(\alpha + \alpha), \quad \alpha_o = Re \alpha. \]
(5.4)

The components of K and the function \( f(\xi, t) \) ought to fulfill the equation
\[ af_t + \beta f_t + (2\alpha - (\alpha + \bar{a})) f = \bar{a} \xi + \bar{b} + iP(t). \]
(5.5)

It should be noticed, whichever the function \( f(\xi, t) \), the metric determined by (5.1), being independent of \( \alpha \), always has the symmetry \( \partial_{\xi} \). This fact, as it should be, is also confirmed by the (5.5) and (5.3) equations.

Determining the general solution of (5.5) it is convenient to distinguish the cases of \( a \) being equal to zero or different from it.

For \( a \neq 0, \)

\[ \text{TABLE III. Symmetries of the Kundt solutions.} \]

<table>
<thead>
<tr>
<th>( f(\xi, t) )</th>
<th>( f_{\xi} \neq 0 )</th>
<th>( G )</th>
<th>Homothetic and isometric killing vectors</th>
<th>( \kappa = \text{const} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \phi(\xi - ib) )</td>
<td>1</td>
<td>( T + bR )</td>
<td>0</td>
<td>( \text{const} )</td>
</tr>
<tr>
<td>( e^{\xi} \phi(\xi - ib) )</td>
<td>1</td>
<td>( \mathcal{A} = T + iD )</td>
<td>( 2l )</td>
<td>( \text{const} )</td>
</tr>
<tr>
<td>( e^{\xi} \xi + \xi' )</td>
<td>2</td>
<td>( T, nL + R )</td>
<td>0</td>
<td>( \text{const} )</td>
</tr>
<tr>
<td>( \xi^2 + 1 + \xi' )</td>
<td>2</td>
<td>( T, \mathcal{A} = (T - (n/2)L) )</td>
<td>( 2l )</td>
<td>( \text{const} )</td>
</tr>
</tbody>
</table>

\( T := \partial_t, \ \mathcal{R} := \partial_t - \partial_\xi, \ D := \xi \partial_t + \bar{\xi} \partial_\xi + r \partial_r, \ L := i \partial_t + i \xi + \bar{\xi} - 2r \partial_r, \)


Salazar I., Garcia D., and Plebański 2194
f = (a\sigma^{-1})^{-1} [\phi (2a - 1) dt] + \xi \sigma^{-1} dt + \frac{1}{2} \int a\sigma^{-1} dt - \int a\sigma^{-1} dt + i \int a\sigma^{-1} dt,
\quad (5.6)

where \sigma(t) = \exp(-a^{-1} dt).

For a = 0,
\begin{equation}
\begin{aligned}
f &= \phi (t) [\alpha_0 \xi^2 + \tau_1] + \frac{n+1}{2} (\xi/\alpha_0) \xi - (\alpha_0 + a_0)^{-1} \\
&\times \left[ \frac{\mathcal{B} + (\xi/\alpha_0) \tau + iP}{\alpha} \right].
\end{aligned}
\end{equation}

An exhaustive list of structural functions \( f(\xi, t) \) and the symmetries determined by them are shown in Table IV. The symmetry \( \sigma_{n} \), the existence of which does not depend on the choice of \( f(\xi, t) \), will be denoted simply by \( K \). Some Killing directions depending on \( \tau \), which in its turn obeys an ordinary linear differential equation of order \( s \), will be designated by \( K(s) \), where \( s \) is now equal to the maximal number of independent constants.

6. Conclusions

For completely arbitrary structural functions the nontwisting type-N solutions, except the \( R \) metric which always possesses one translation, have no symmetries.

The nontwisting type-N solutions with nonvanishing cosmological constant allow the existence of isometries [at most two Killing vectors] if the structural function \( f(\xi, t) \) has the form given by formula (2.7) with \( \kappa \) equal to zero for the \( NT(\mathcal{L}, Z, \varepsilon) \) metrics, and by (3.6) for the \( K(\alpha) \) solutions; by the use of coordinate gauges one brings the structural function \( f(\xi, t) \) to the form shown in Table I \( (\kappa = 0) \) and II, respectively. The vacuum \( NT(0, Z, \varepsilon) \) and Kundt solutions with \( f(\xi, t) \)

**TABLE IV.** Symmetries of the Robinson solutions.

<table>
<thead>
<tr>
<th>( f(\xi, t) )</th>
<th>( f_{\xi_{\xi}} \neq 0 )</th>
<th>( G )</th>
<th>Conformal, homothetic, and isometric Killing vectors</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \phi(\xi, t) )</td>
<td>1</td>
<td>1</td>
<td>( K )</td>
</tr>
<tr>
<td>( \phi(\xi^2) )</td>
<td>2</td>
<td>2</td>
<td>( K, T - nR )</td>
</tr>
<tr>
<td>( t^{-1} \phi(t^{-1}) )</td>
<td>2</td>
<td>2</td>
<td>( K, \varepsilon = C )</td>
</tr>
<tr>
<td>( t^{-1+n} \phi(t^{n}) )</td>
<td>2</td>
<td>2</td>
<td>( \varepsilon = C + aH - nT + bR )</td>
</tr>
<tr>
<td>( \frac{\xi + \mu(t^2 - n)^{1/2}}{\xi + \mu(2 + \mu \sqrt{n})} )</td>
<td>2</td>
<td>2</td>
<td>( \varepsilon = 1 - \mu )</td>
</tr>
<tr>
<td>( t^{n} \exp \xi )</td>
<td>2</td>
<td>2</td>
<td>( M(\mu) = L = (1/n^{1/2} + 2 + \mu \sqrt{n}) )</td>
</tr>
<tr>
<td>( \exp \xi )</td>
<td>3</td>
<td>2</td>
<td>( K, M(\mu) = 0, T )</td>
</tr>
<tr>
<td>( i\alpha(t) \ln \xi, A(t) \ln \xi )</td>
<td>2</td>
<td>2</td>
<td>( K, R + 2K )</td>
</tr>
<tr>
<td>( (t^{-2} - n)^{1/2} \ln \xi )</td>
<td>3</td>
<td>2</td>
<td>( K, R, \varepsilon = C - nT - nK )</td>
</tr>
<tr>
<td>( i\alpha(t) \ln \xi )</td>
<td>3</td>
<td>2</td>
<td>( K, R, \varepsilon = C - nT - nK )</td>
</tr>
<tr>
<td>( \phi(t) \xi^{2} )</td>
<td>6</td>
<td>2</td>
<td>( K, K(4), \varepsilon = H )</td>
</tr>
<tr>
<td>( \frac{i\xi^{2}}{(t^{-2} - n)} \frac{\xi + \mu(t^{2} - n)^{1/2}}{\xi + \mu(2 + \mu \sqrt{n})} )</td>
<td>7</td>
<td>2</td>
<td>( K, K(4), \varepsilon = 1 - \mu )</td>
</tr>
<tr>
<td>( t^{1+n} \exp 2ikt )</td>
<td>7</td>
<td>2</td>
<td>( K, K(4), T - kR )</td>
</tr>
<tr>
<td>( t^{-2-2n} \xi^{2} )</td>
<td>7</td>
<td>2</td>
<td>( K, K(4), L - kR )</td>
</tr>
<tr>
<td>( t^{2-2n} \exp 2ikt )</td>
<td>7</td>
<td>2</td>
<td>( K, K(4), \varepsilon = aH )</td>
</tr>
<tr>
<td>( K = \partial_{t}, T = \partial_{t}, D_{t} = \xi_{t} \partial_{t} + \xi_{t} \partial_{t}, L_{t} = \partial_{t} - \xi_{t}, H_{t} = D + 2\tau K, C_{t} = t^{2}T + tD - 2\xi K )</td>
<td>7</td>
<td>2</td>
<td>( K, K(4), \varepsilon = 1 - \mu )</td>
</tr>
</tbody>
</table>

2195  
Salazar I., García D., and Plebański 2195

2195  

This article is copyrighted as indicated in the article. Reuse of AIP content is subject to the terms at: http://scitation.aip.org/termsconditions. Downloaded to IP: 148.247.185.21 On: Fri, 18 Apr 2014 18:04:30
given correspondingly by formulas (2.7) and (4.6) permit maximally the existence of two Killing vectors or one homothetic and one Killing vector; see also Tables I and III, respectively. As is well known the type-N Robinson solutions permit the existence of six Killing vectors for some specific structural functions $f(\xi, t)$. The maximal group order for the conformal Killing vectors is seven, which is in agreement with the results by Collinson and French. The structural functions, which allow symmetries of the $R$ metric, are given by formulas (5.6) and (5.7); a list of representative structural functions with their symmetries are shown in Table IV.

ACKNOWLEDGMENT

One of us (H.S.I.) acknowledges the Consejo Nacional de Ciencia y Tecnología for financial assistance through a doctoral fellowship.