Some solutions of complex Einstein equations
J. F. Plebański

Citation: Journal of Mathematical Physics 16, 2395 (1975); doi: 10.1063/1.522505
View online: http://dx.doi.org/10.1063/1.522505
View Table of Contents: http://scitation.aip.org/content/aip/journal/jmp/16/12?ver=pdfcov
Published by the AIP Publishing

Articles you may be interested in

Some solutions of Einstein's equations with shock waves

Some exact inhomogeneous solutions of Einstein's equations with symmetries on the hypersurfaces t=const

The investigation of some selfsimilar solutions of Einstein's equations

Some solutions of the Einstein field equations for a rotating perfect fluid
J. Math. Phys. 16, 125 (1975); 10.1063/1.522404

Complex 2Form Representation of the Einstein Equations: The Petrov Type III Solutions
1. NOTATION, CONVENTIONS AND TERMINOLOGY

The fact that the large family of type D solutions of Einstein—Maxwell equations\(^1\) can be explicitly exhibited as a real cross section of a complex double Kerr—
Schild metric and that its generalization derived with Demiański\(^2\) has the same property, has stimulated the
author's interest in (i) the complex Riemannian geometry
and (ii) the double Kerr—Schild metrics as such. The
second subject will be extensively studied in a forth­
coming publication by Schild and the present author.
This article is basically intended as an outline of some
general facts and results concerned with the complex
Riemannian geometry as such.

The complex four-dimensional Riemannian space is
a pair consisting of a four-dimensional differential
analytic manifold \(M_4\) and (with \(e^a \in \Lambda^1,\ a = 1, 2, 3, 4\) the
metric given by
\[
V_A; ds^2 = 2e^a e^{\bar{a}} + 2e^{\bar{a}} e^a = \kappa_{a\bar{a}} e^a e^{\bar{a}} \in \Lambda^2 \otimes \Lambda^2.
\]
(1.1)

The tetradial indices—the forms \(e^a\) define a null tetrad—
are to be manipulated by
\[
(g_{a\bar{a}}) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.
\]
(1.2)

The Pauli matrix is a \(2 \times 2\) matrix with entries in \(\Lambda^1\) and
connects \(\Lambda^1\) with spinorial objects
\[
(g_{a\bar{a}}) \epsilon = \sqrt{2} \begin{pmatrix} e^1 & e^2 \\ e^3 & e^4 \end{pmatrix}.
\]
Thus
\[
ds^2 = -\det(g_{a\bar{a}}) = \frac{1}{2} \kappa_{a\bar{a}} e^a e^{\bar{a}} = ds^2.
\]
(1.4)

Consider now two independent sets of \(2 \times 2\) complex
matrices with determinant equal to one:
\[
t^a_A \in SL(2, C) - \det(t^a_A) = 1,
\]
\[
t^{\bar{a}}_B \in SL(2, C) - \det(t^{\bar{a}}_B) = 1.
\]
(1.5)

It is now obvious that \(e^a\) defined by
\[
\sqrt{2} \begin{pmatrix} e^1 & e^2 \\ e^3 & e^4 \end{pmatrix} = (t^a_A \bar{t}^{\bar{b}}_B) e^a e^{\bar{b}}
\]
still gives \(2e^a e^{\bar{a}} + 2e^{\bar{a}} e^a = ds^2\). Thus, we can consider
as the tetradial gauge group
\[
\mathcal{G} = SL(2, C) \times SL(2, C).
\]
(1.7)

[More precisely, one could work with the tetradial
gauge imposing on the \(2 \times 2\) matrices only the condition
\[
det(t^a_A) \det(t^{\bar{a}}_B) = 1; \text{ the obvious group of the tetradial}
geasure in \(V_A, O(4, C), \text{ decomposes into the product}
O(3, C) \otimes O(3, C); \text{ this fact permits us to identify the}
tetradial gauge group with \(\mathcal{G}.\)]

Although we are interested in the complex geometry,
we can proceed with the standard conventions of the
spinor calculus, transforming respectively undotted and
dotted indices by objects from \(SL(2, C)\) and \(SL(2, C)\),
forgetting however about the condition \((t^a_A) \star = \bar{t}^{\bar{a}}_B\); from the case of real \(V_A\) of the signature \((++, +,-),\) which
causes \(\mathcal{G}\) to reduce to \(SL(2, 2; C)\) homomorphic with \(SO(3, 1, R)\).

[In the "real" case, the permitted gauge transformations must conserve the condition \((e^a)^* = e^a;\) in the com­
plex geometry where this condition is abandoned,
matrices \(SL(2, C)\) and \(SL(2, C)\) are independent.]

The equation
\[
g^{\bar{a}a} = S_{\bar{a}a} \omega_{\bar{a}a} = S_{\bar{a}a} e^{\bar{a}} e^a + \omega_{\bar{a}a} \bar{e}^a e^{\bar{a}}
\]
defines in \(\Lambda^2\) the two forms of the spin tensor:
\[
S_{\bar{a}a} = \frac{1}{2} \kappa_{\bar{a}a} e^\bar{a} e^a + \frac{1}{2} \kappa_{\bar{a}a} e^\bar{a} e^a
\]
\[
\omega_{\bar{a}a} = \frac{1}{2} \kappa_{\bar{a}a} e^\bar{a} e^a + \frac{1}{2} \kappa_{\bar{a}a} e^\bar{a} e^a
\]
(1.8)

Explicitly, we have:
\[
S^{12} = 2e^1 \wedge e^2, \quad S^{12} = 2e^1 \wedge e^2 + 2e^2 \wedge e^2, \quad S^{22} = 2e^2 \wedge e^2,
\]
\[
S^1 = 2e^1 \wedge e^1, \quad S^2 = 2e^2 \wedge e^2.
\]
(1.9)

The forms \(S^{ab}\) and \(S^{\bar{a} \bar{b}}\) are respectively self-dual
and anti-self-dual under Hodge's star operation:
\[
* S^{ab} = S^{\bar{a} \bar{b}}, \quad * S^{\bar{a} \bar{b}} = -S^{ab}.
\]
(1.10)

[The duality operation \(*\) acting on \(\omega = \Lambda^2,\]
\[
\omega = \frac{1}{p!} \omega_{a_1 \cdots a_p} e^{a_1} \wedge \cdots \wedge e^{a_p}
\]
maps it into
\[
* \omega = \frac{1}{p!} p! \exp \frac{i\gamma}{2} (p') - 2e^{a_1} \wedge \cdots \wedge e^{a_p}
\]
(1.11)

where indices of the Levi—Civita symbol \(e^{a_1} \cdots e^{a_p}\)
are manipulated by \((g^{ab})\) and \(p' = 4 - p;\) this definition is so
constructed that \(* * \omega = \omega \) for every \(\omega.\]
Now, the connection forms $\Gamma_{ab}^c = \Gamma_{1ab}^c \in A^1$ are defined by the first structure equations

$$de^c = e^d \wedge \Gamma_{db}^c$$  \hspace{1cm} (1.14)$$

and if $\partial_a$ is the inverse tetrad (acting on any scalar $T$: $(d - e^aR_a)T = 0$), then the Ricci rotation coefficients $\left(\Gamma_{abc}^d\right) = \Gamma_{1abc}^d$ can be computed from the commutators

$$\partial_a \partial_b \partial_c = [\Gamma_{ab}^e, \Gamma_{ec}^f] = \Gamma_{1ab}^e \Gamma_{1ec}^f - \Gamma_{1eb}^e \Gamma_{1ac}^f.$$  \hspace{1cm} (1.15)

The objects $\Gamma_{1ab}$ are equivalent to their spinorial images

$$\Gamma_{AB} := -\frac{1}{4} \Gamma_{1ab} E_{AB} \equiv \Gamma_{1AB}^C E_{CA}$$

$$\Gamma_{a} := -\frac{3}{4} \Gamma_{1ab} T_a.$$  \hspace{1cm} (1.16)

The explicit form of these relations is

$$\left(\Gamma_{AB}\right) := -\frac{1}{2} \left( \begin{array}{cccc} 2 \Gamma_{42} & \Gamma_{12} + \Gamma_{34} & \\
\Gamma_{12} & 2 \Gamma_{34} & \\
\Gamma_{34} & 2 \Gamma_{42} & \\
& & 2 \Gamma_{34} & \\
\end{array} \right)$$

$$\left(\Gamma_{A}^{\prime}\right) := -\frac{1}{2} \left( \begin{array}{cccc} 2 \Gamma_{11} & -\Gamma_{12} & \\
-\Gamma_{12} & 2 \Gamma_{11} & \\
& & 2 \Gamma_{11} & \\
& & & 2 \Gamma_{11} & \\
\end{array} \right).$$  \hspace{1cm} (1.17)

and under the tetradial gauge $\zeta$ from (1.7) the connections transform according to

$$\Gamma^{\alpha^e}_{\beta} = \zeta^{\alpha^e}_{\beta} + \zeta^{\alpha^e}_{\gamma} \zeta^\gamma_\beta \Gamma^{\gamma}_{\delta} + \zeta^\gamma_\beta \zeta^{\gamma^e}_{\delta}$$

$$\zeta^{\alpha^e}_{\gamma} := \frac{1}{2} \Gamma^{\alpha^e}_{\beta} \Gamma^{\beta}_{\delta} \Gamma^{\gamma}_{\delta}$$

$$\zeta^\gamma_\beta := \frac{1}{2} \zeta^\gamma_\beta \zeta^{\gamma^e}_{\delta} \Gamma^{\delta}_{\delta}.$$  \hspace{1cm} (1.18)

This fact will be of crucial importance in our further considerations: (1.18) exhibits explicitly the (irreducible) decomposition of the transformational properties of $\Gamma$s which occurs on the level of the spinorial images and which corresponds to the independent factors in the (cross) factors (1.7) for the gauge group.

Notice that while

$$d\xi^a \wedge \Gamma^a \wedge \Gamma^a,$$  \hspace{1cm} (1.19)

then

$$d\xi^a \wedge \Gamma^a \wedge \Gamma^a,$$  \hspace{1cm} (1.20)

consider now the two spinorial curvature forms

$$R^{A}_{B} := d\Gamma_{B}^{A} + \Gamma_{B}^{C} \wedge \Gamma_{C}^{A}$$

$$= -\frac{1}{2} C^{A}_{B,C,D} S^{C,D} = \frac{1}{2} C^{A}_{B,C,D} S^{C,D}.$$  \hspace{1cm} (1.21a)

$$R^{\alpha^e}_{\beta} := d\Gamma^{\alpha^e}_{\beta} + \Gamma^{\alpha^e}_{\gamma} \wedge \Gamma^{\gamma}_{\beta}$$

$$= -\frac{1}{2} C^{\alpha^e}_{\beta,C,D} S^{C,D} = \frac{1}{2} C^{\alpha^e}_{\beta,C,D} S^{C,D}.$$  \hspace{1cm} (1.21b)

The equalities in (1.21) correspond to the Cartan structure equations $d\Gamma_{B}^{A} + \Gamma_{B}^{C} \wedge \Gamma_{C}^{A}$, with $R_{abcd}$ being the tetrad components of the Riemann tensor. The symbols used in (1.21) are defined as follows: Let

$$R_{ab} := R_{ab}^{a} = \text{Ricci tensor}, \quad R = R_{ab}^{a} = \text{scalar curvature},$$

and let $C_{abcd} = R_{abcd}$ be the conformal curvature. Then we have

$$C_{abcd} = \frac{1}{2} C_{abc} e^a \wedge e^b \wedge e^c \wedge e^d,$$

$$C_{abcd} = \frac{1}{2} C_{abc} e^a \wedge e^b \wedge e^c \wedge e^d.$$  \hspace{1cm} (1.22)

Thus, it makes sense to talk about the conformal curvature of a complex $V_4$ as being of the types $[1-1-1-1], [2-2], [2-1-1], [-], [4], [3-1], [1-1-1-1], [2-2], [2-1-1], [-], [4], [3-1].$ (1.28)

This article is copyrighted as indicated in the article. Reuse of AIP content is subject to the terms at: http://scitation.aip.org/termsconditions. Downloaded to IP: 148.247.185.21 On: Fri, 18 Apr 2014 18:19:56
the Debever vectors on the case considered. Let \( K^{a} \) be a
null vector; then it can always be considered as induced
by a pair of spinors of both types:

\[ K^{a} \, dx^{a} = -\frac{1}{2} g^{ab} K_{b} \tilde{K}^{b}. \]  

(1.29)

The object

\[ K_{ab} \delta \beta = \delta (K_{a}) C_{a b} \delta \beta = K^{a}_{b} K_{a} C_{b} \delta \beta^{a}_{b} K_{a} K^{b} \]  

(1.30)

has all symmetries of the conformal curvature tensor
and hence is characterized entirely by its two spinorial
images analogous to \( C_{ABCD} \) and \( \tilde{C}_{ABCD} \) which can
be easily computed as given by

\[ K_{ABCD} = 3 K_{A} K_{B} K_{C} K_{D} - \sum_{K=PSR} R^{K} R^{K} R^{K} \]  

(1.31)

It easily follows that the linear operation \( \tilde{J}(K_{a}) \) defined
by (1.30) is nil-potent:

\[ \tilde{J}^{2}(K_{a}) = 0. \]  

(1.32)

Moreover, one easily infers that

\[ \tilde{J}(K_{a}) C_{a b} \delta \beta = 0 \]  

(1.33)

is the necessary and sufficient condition for both \( K_{a} \) and
\( K_{b} \) to be generic \( P \)-spinors of both types. We will thus
call the null vectors defined by (1.29), with both \( K_{a} \) and
\( K_{b} \) being generic \( P \)-spinors, the generalized DP
(Debever-Penrose) vectors. In a general complex \( V_{4} \) all
of them are complex. The number of different (in the
sense of the linear independence in pairs) DP vectors
depends on the type of \( V_{4} \); e.g., for the type \( G \otimes G \) there
exist 16 such vectors, while for the type \( N \otimes N \) we
have only one DP vector. The spaces of the type [something] \( \otimes [ - ] \) determine only the spinor \( K_{a} \) in DP vector;
the spinor \( \tilde{K}^{a} \) can be here selected completely
arbitrarily.

Notice that the concept of the generalized DP vector
applies mutatis mutandi also in the case of a real \( V_{4} \) of
the signature \( (++++) \) where the \( P \)-spinors of the
second type become complex conjugates of the \( P \)-spinors
of the first type. Thus, with the conformal curvature of
the type \( G \) one can construct precisely 16 such complex
vectors; the 4 real vectors among these 16 objects
will be the standard DP vectors. Of course, there will
be numerous additional relations among the discussed
16 vectors.

Now, Newman and Penrose call such \( ^{6} \) complex \( V_{4} \)’s
which have in our notation \( \tilde{C}^{ABCD} = 0 \) “right-flat heaven”
and if \( C_{ABCD} = 0 \), correspondingly, “left-flat heaven.”

Following them (in a sense), but insisting on a more
contrasting (perhaps one could say—“manichean”) terminology,
we propose to call all objects which are
\( SL(2, \mathbb{C}) \) scalars and are geometric objects with respect
to \( SL(2, \mathbb{C}) \), the “heavenly objects.” Parallely the objects
which with respect to \( SL(2, \mathbb{C}) \) are scalars and are
geometric objects with respect to \( SL(2, \mathbb{C}) \), will be the
“hellish objects.” The objects whose components mix
in an irreducible fashion under \( SL(2, \mathbb{C}) \) and \( SL(2, \mathbb{C}) \)
transformations will be then the “earthly objects.” An
absolute scalar with respect to \( \zeta = SL(2, \mathbb{C}) \times SL(2, \mathbb{C}) \)—
in a degenerate sense because it cannot be in heaven and
hell simultaneously—we assign to earth.

Therefore, the objects discussed up to now, can be
classified according to Table I.

<table>
<thead>
<tr>
<th>Hellish</th>
<th>Earthly</th>
<th>Heavenly</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \delta^{AB} )</td>
<td>( \delta^{AB} )</td>
<td>( \delta^{AB} )</td>
</tr>
<tr>
<td>( \Gamma^{AB} )</td>
<td>( \Gamma^{AB} )</td>
<td>( \Gamma^{AB} )</td>
</tr>
<tr>
<td>( C_{ABCD} )</td>
<td>( C_{ABCD} )</td>
<td>( C_{ABCD} )</td>
</tr>
</tbody>
</table>

TABLE I.

The space \( \tilde{C}^{ABCD} = 0 \) we will call “weak heaven”;
heaven”; we permit here for \( C_{a} \) and \( R \) being \# \#; a complex
space \( V_{4} \) where there exists such a choice for the
null tetrad that

\[ \tilde{R}^{AB} = 0 \]  

(1.34)

we will call the “strong heaven.”

With these “hellish” objects vanishing we have a for­
eteriori from (1.21b)

\[ C^{ABCD} = 0, \quad C_{ABCD} = 0, \quad R = 0. \]  

(1.35)

Therefore, with \( G_{a} = R_{a} = \frac{1}{2} g_{ab} R \) we conclude that the
Einstein vacuum equations (in the complex \( V_{4} \))

\[ G_{a} = 0 \]  

(1.36)

are automatically fulfilled in the “strong heaven.” Of
course, “strong heaven” is nontrivial if \( C_{ABCD} = 0 \);
otherwise \( V_{4} \) is flat.

2. HEAVENLY TETRAD AND FIRST HEAVENLY
EQUATION

Assuming (1.34), it follows from (1.20) that

\[ dS^{AB} = 0. \]  

(2.1)

Thus (in a simply connected region) there exists such
\( U^{AB} \in A_{4} \) that

\[ S^{AB} = dU^{AB}. \]  

(2.2)

Using the explicit form of \( S^{AB} \) from (1.10), we infer
now that

\[ 2e^{A} \wedge e^{B} = dq^{A} \wedge dq^{B}, \quad 2e^{A} \wedge e^{B} = dq^{A} \wedge dq^{B}, \]  

(2.3)

so that \( dq^{A} \wedge dq^{B} = 0 = dq^{A} \wedge dq^{B} \). Thus, by applying the
Darboux theorem in our complex \( V_{4} \), we deduce the
existence of such scalars \( p, q, r, s \) that

\[ 2e^{A} \wedge e^{J} = 2 dp \wedge dq = 2 dp \wedge dq + dr \]  

\[ 2e^{A} \wedge e^{B} = 2 dr \wedge ds = 2 dr \wedge ds + d) \]  

- \( dv = e^{A} \wedge e^{B} \wedge e^{C} \wedge e^{D} \) 

\[ = dp \wedge dq \wedge dr \wedge ds. \]  

(2.4)

It follows that (i) \( \{ x^{a} \} = \{ pqrs \} \) can be used as the in­
dependent coordinates and (ii) that the heavenly tetrad
is given in terms of these coordinates by

\[ e^{A} = A dp + B dq, \quad e^{A} = Ed + F ds, \]  

\[ e^{4} = C dp + D dq, \quad e^{2} = G dr + H ds. \]  

(2.5)

J.F. Plebański
Entering with it into (2.4) we infer that the structural functions must fulfill two conditions

\[
AD - BC = 1, \quad \text{(2.6a)}
\]

\[
EH - FG = 1. \quad \text{(2.6b)}
\]

We still must assure that \( S_1 = -e^1 \wedge e^2 + e^3 \wedge e^4 \) be closed which requires

\[
d(-e^1 \wedge e^2 + e^3 \wedge e^4)
= -d[(AG - CE) dp \wedge dr + (AH - CF) dp \wedge ds]
+ (BG - DE) dq \wedge dr + (BH - DF) dq \wedge ds = 0. \quad \text{(2.7a)}
\]

This condition is equivalent to the equations

\[
(AG - CE)_p = (BG - DE)_p = 0, \quad (AG - CE)_q - (AH - CF)_q = 0, \quad (AH - CF)_p - (BG - DE)_p = 0. \quad \text{(2.7b)}
\]

Last equations can be readily seen to imply the existence of a function \( \Omega = \Omega^{pqrs} \) — called subsequently the first key function — such that

\[
AG - CE = \Omega_{pr}, \quad BG - DE = \Omega_{qr}, \quad AH - CF = \Omega_{ps}, \quad BH - DF = \Omega_{qs}. \quad \text{(2.7c)}
\]

We then easily see that

\[
S_1 = d(\Omega_{pr} dp + \Omega_{qr} dq) = -d(\Omega_{ps} dr + \Omega_{qs} ds) \quad \text{(2.8)}
\]

so that \( dS_1 = 0 \) is assured.

Considering now (2.7c) as four equations on \( EFGH \) with \( ABCD \) assumed known, one easily solves them, using (2.4a) and obtaining

\[
E = B\Omega_{pr} - A\Omega_{qr}, \quad G = C\Omega_{ps} - B\Omega_{qs}, \quad F = B\Omega_{ps} - A\Omega_{qs}, \quad H = C\Omega_{pr} - B\Omega_{qr}. \quad \text{(2.9)}
\]

This substituted into (2.6b) yields for the key function \( \Omega \) the first heavenly equation

\[
\begin{pmatrix}
\Omega_{pr}
\\Omega_{qr}
\end{pmatrix}
\begin{pmatrix}
p
q
\end{pmatrix}
= 1. \quad \text{(2.10)}
\]

Of course, \( ABC \) and \( D \) cancel out here because of (2.6a); quite similarly, if we substitute our tetrad into \( ds^2 \) with \( EFG \) and \( H \) of the form (2.9), we obtain

\[
V_4 : ds^2 = 2\Omega_{pr} dp dr + 2\Omega_{qr} dq ds + 2\Omega_{ps} dq dr + 2\Omega_{qs} dq ds. \quad \text{(2.11)}
\]

Therefore, we have demonstrated that the most general "strong heaven" \( V_4 \) is determined by one key function \( \Omega \) which fulfills (2.10) and defines \( g_{\mu\nu} \) by (2.11). Observe that with coordinates ordered as \([s^2] = \{pqrs\}\),

\[
(g_{\mu\nu}) = \begin{pmatrix}
0 & 0 & \Omega_{pr} & \Omega_{ps} \\
0 & 0 & \Omega_{qr} & \Omega_{qs} \\
\Omega_{pr} & \Omega_{qr} & 0 & 0 \\
\Omega_{ps} & \Omega_{qs} & 0 & 0
\end{pmatrix},
\]

and

\[
\det(g_{\mu\nu}) = 1. \quad \text{(2.13)}
\]

For the arbitrary scalar \( \phi \) we have thus

\[
\phi^{\sigma}_{\mu} = \frac{1}{\sqrt{g}} g^{\mu\nu} \partial_{\nu} \phi = \frac{1}{\sqrt{g}} \left( \frac{\partial_{\mu}\Omega_{pqrs} - \Omega_{pqrs} \partial_{\mu}\phi}{\Omega_{pqrs}} \right) - \Omega_{pqrs} \partial_{\mu}\phi
+ \Omega_{pqrs} \partial_{\mu}\phi
\quad \text{(2.14)}
\]

It follows that if we interpret \( \Omega \) as a scalar, then

\[
\Omega^{\sigma}_{\mu} = 4. \quad \text{(2.15)}
\]

Now, it is clear that \( ABC \) and \( D \) in the heavenly tetrad—with \( EFG \) and \( H \) interpreted according to (2.9)—just correspond to the residual freedom of \( SL(2,C) \) gauge. Thus, not losing generality but only taking a definite choice for this gauge, we can in particular assume

\[
A = 1 = D, \quad B = 0 = C. \quad \text{(2.16)}
\]

This gives for the heavenly tetrad

\[
e^1 = dp, \quad e^2 = - dq, \quad e^3 = \Omega_{pr} dr + \Omega_{qr} ds, \quad e^4 = - \Omega_{ps} dr + \Omega_{qs} ds. \quad \text{(2.17)}
\]

The "heavenly" connections are given by

\[
\alpha = \Gamma_{\mu s} = - K dp + L dq, \quad \beta = \Gamma_{\mu s} = - L dp + M dq, \quad \gamma = \Gamma_{\mu s} = - N dp + K dq \quad \text{(2.19)}
\]

where

\[
K = \frac{\partial_\mu \Omega_{pqrs} - \Omega_{pqrs} \partial_\mu \phi}{\Omega_{pqrs}}, \quad L = - \frac{\partial_\mu \Omega_{pqrs} - \Omega_{pqrs} \partial_\mu \phi}{\Omega_{pqrs}}, \quad M = \frac{\partial_\mu \Omega_{pqrs} - \Omega_{pqrs} \partial_\mu \phi}{\Omega_{pqrs}}
\]

The identities in (2.20) follow from (2.10); formulas with logarithms apply if one assumes \( \Omega_{\mu\nu} \neq 0 \).

Of course, (2.18) already assures that the \( V_4 \) under
study, being "strong heaven" has $C_{qy}=0$ and $C_{ABCD}$ = 0. Now, the curvature quantities $C_{ABCD}$ or equivalently $C^{(v)}$, $a=1, \ldots, 5$ can be computed from (1.21a) which specialized on the present case amounts to

$$\begin{align*}
d\beta + 2\beta \wedge \alpha \\
d\alpha + \beta \wedge \gamma \\
d\gamma + 2\alpha \wedge \gamma
\end{align*}$$

Using the symbol $dV$ from (2.4), we have thus

$$\begin{align*}
(C^{(5)}) & \quad dV = -2e^3 \wedge e^1 \\
(C^{(4)}) & \quad dV = \begin{pmatrix} d\beta + 2\beta \wedge \alpha \\ d\alpha + \beta \wedge \gamma \\ d\gamma + 2\alpha \wedge \gamma \end{pmatrix} \\
(C^{(3)}) & \quad dV = -2e^3 \wedge e^1 \\
(C^{(2)}) & \quad dV = -2e^3 \wedge e^1 \\
(C^{(1)}) & \quad dV = -2e^3 \wedge e^1
\end{align*}$$

Substituting here (2.19) and (2.17), one easily finds that

$$\begin{align*}
\frac{1}{2}C^{(5)} &= \tilde{\varepsilon}_2 M, \quad \frac{1}{2}C^{(4)} = \tilde{\varepsilon}_2 L, \\
\frac{1}{2}C^{(3)} &= \tilde{\varepsilon}_2 K = -\tilde{\varepsilon}_2 L, \\
\frac{1}{2}C^{(2)} &= -\tilde{\varepsilon}_2 K, \quad \frac{1}{2}C^{(1)} = -\tilde{\varepsilon}_2 N.
\end{align*}$$

3. AN ALTERNATIVE FORM OF THE HEAVENLY TETRAD AND THE SECOND HEAVENLY EQUATION

Although the description of the heavenly tetrad in terms of $\Omega$ satisfying (2.10) is symmetric and simple enough, we are able to provide a still simpler—and more convenient for many purposes—alternative description of this tetrad. Let

$$x=\Omega_p, \quad y=\Omega_q. \quad (3.1)$$

Then Eq. (2.10) takes the form (forgetting for a moment about coordinates $p$ and $q$)

$$\frac{\delta(x,y)}{\delta(r,s)} = 1, \quad (3.2)$$

so that certainly $x$ and $y$ can be used as new coordinates in the place of $r$ and $s$. The tetrad (2.17) is now given by

$$\begin{align*}
e^1 &= dp, \quad -e^1 = dx + A dp + B dq, \\
e^2 &= dq, \quad -e^2 = dy + B dp + C dq,
\end{align*}$$

where in terms of $\Omega$ we have

$$A = -\Omega_{pp}, \quad B = -\Omega_{pq}, \quad C = -\Omega_{qq}. \quad (3.4)$$

We will now consider the heavenly tetrad (3.3) *prima facie* understanding $A$, $B$, and $C$ as the three structural functions given in coordinates $\{p_{\nu}\} = \{pqxy\}$. The forms $S_{
u}^\mu$ induced by (3.3) must be all closed. The forms $S_{\nu}^1 = 2e^3 \wedge e^1 + 2dp \wedge dq$ and $S_{\nu}^2 = -e^1 \wedge e^3 + e^1 \wedge e^5$ are closed independently of the shape of $A$, $B$, and $C$. Thus, with

$$\begin{align*}
\frac{1}{2}S_{\mu}^{33} &= e^3 \wedge e^3 + dx \wedge dq + (AC - B^3) dp \wedge dq \\
&\quad + A dp \wedge dq + B(dx \wedge dp + dq \wedge dy) + C dx \wedge dq
\end{align*}$$

we obtain from $dS_{\mu}^{33}$ = 0 the conditions

$$\begin{align*}
A_x + B_y &= 0, \quad (3.6a) \\
B_x + C_y &= 0, \\
(AC - B^3)_x + B_y - C_z &= 0, \\
(AC - B^3)_y - A_x + B_p &= 0. \quad (3.6b)
\end{align*}$$

One easily shows that (3.6a) implies (and is implied by) the existence of a function $\theta = \theta(pqxy)$ such that

$$A = -\Theta_{yy}, \quad B = \Theta_{xy}, \quad C = -\Theta_{xx}. \quad (3.7)$$

But this substituted into (3.6) gives

$$\begin{align*}
0_r &= \left(\Theta_{xx} \Theta_{yy} - (\Theta_{xy})^2 + \Theta_{xy} \theta_y \right) = 0.
\end{align*}$$

Consequently, $\Theta_{xx} \Theta_{yy} - (\Theta_{xy})^2 + \Theta_{xy} \theta_y$ is a function of $f(p,q)$, with $f(p,q)$ being arbitrary. Therefore, introducing $\Theta = \theta - xy$ we arrive at the conclusion that with:

$$A = -\Theta_{yy}, \quad B = \Theta_{xy}, \quad C = -\Theta_{xx}, \quad (3.9)$$

and the second key function $\Theta = \Theta(pqxy)$ fulfilling the second heavenly equation

$$\Theta_{xx} \Theta_{yy} - (\Theta_{xy})^2 + \Theta_{xy} \theta_y = 0. \quad (3.10)$$

The forms $S_{\nu}^\mu$ are all closed. [Indeed, (3.9) used in (3.5) gives

$$\frac{1}{2}S_{\mu}^{33} = e^3 \wedge e^3 = dx \wedge dq + d\Theta_{yy} dp \wedge dq + dq \wedge d\Theta_{xx}. \quad (3.11)$$

The heavenly metric given in the terms of the present coordinates and the second key function is thus

$$V_4: \quad ds^2 = 2dp(dx - \Theta_{yy} dp + \Theta_{xy} dq) + 2dq(dy + \Theta_{xx} dp - \Theta_{xy} dq). \quad (3.12)$$

For the natural tetrad we have now

$$\begin{align*}
e^1 &= dp, \quad -e^1 = dq, \\
e^2 &= dx - \Theta_{xx} dp + \Theta_{xy} dq, \quad e^2 = dy + \Theta_{yy} dp - \Theta_{xy} dq, \\
e^3 &= dy - \Theta_{yy} dp + \Theta_{xx} dq, \quad -e^3 = dx - \Theta_{xx} dp - \Theta_{xy} dq,
\end{align*}$$

(3.13)

Now, in the present coordinates we have for the metric tensor and its inverse

$$\begin{align*}
(\tilde{g}_{\mu\nu}) &= \begin{pmatrix} 2\Theta_{xx} & 0 & 0 & 0 \\
0 & 2\Theta_{yy} & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \end{pmatrix}, \\
det(\tilde{g}_{\mu\nu}) &= 1.
\end{align*}$$

Knowing $\tilde{g}_{\mu\nu}$ we find that, for any scalar $\phi$,
This article is copyrighted as indicated in the article. Reuse of AIP content is subject to the terms at: http://scitation.aip.org/termsconditions. Downloaded to 148.247.185.21 On: Fri, 18 Apr 2014 18:19:56
Suppose now that we require additionally that $C_{35} = C_{36} = 0$ permitting only for $C_{34} = - C_{35} = a = \text{const}$, to be $\neq 0$. If in fact $a \neq 0$, then the structural functions amount to

$$
\rho = \frac{1}{2} a u^2 - u \Lambda \omega (q r) + c_4 (q r), \quad Q = - \frac{1}{2} a u^2 + v \Lambda \omega (q r) - c_2 (q r),
$$

where $a \neq 0$ and $\Lambda \omega (q r)$, $c_4 (q r)$, $c_2 (q r)$ are arbitrary. The only nontrivial curvature quantities are here given by

$$
C_{34} = - C_{35} = a, \quad \frac{1}{2} C^{(2)} = - \Lambda, \quad \frac{1}{2} C^{(1)} = (u e^{\beta} - v e^{\gamma} - v e^{\gamma} e^{\alpha} - 2 a u v) \Lambda,
$$

$$
+ a (u c_2 + v c_1) + c_3 r - \Lambda \omega (q r) - \Lambda c_2 r - \Lambda c_2 r.
$$

In the limit $a = 0$, $c_4 = 0 - c_2$ and the space in question coincides with (4.9); the curvature quantities

$$
\frac{1}{2} C^{(2)} = - \Lambda, \quad \frac{1}{2} C^{(1)} = (u e^{\alpha} - v e^{\beta} - v e^{\gamma} e^{\alpha}) \Lambda
$$

are considerably more symmetric than the equivalent set in Eq. (4.6).

If one requires in (4.18) more strongly $C_{34} = 0$, then the space becomes a strong heaven. The corresponding structural functions are now given by

$$
\rho = - u \left( \Lambda + \frac{\dot{F}}{F - G} \right) + c_1 + \frac{\dot{F}}{F - G} e^\Lambda \cdot e^u, \quad Q = u \left( \Lambda + \frac{\dot{G}}{F - G} \right) - c_2 - \frac{\dot{G}}{F - G} e^\Lambda \cdot e^u,
$$

where $F = F (q r)$, $G = G(q r)$, $\Lambda = \Lambda (q r)$, $c_1 = c_4 (q r)$, and $c_2 (q r)$ are arbitrary. The curvature is characterized here by

$$
\frac{1}{2} C^{(2)} = - \Lambda^2,
$$

and some $C^{(1)}$; even when $\Lambda^2 \neq 0$, the functions $c_1$ and $c_2$ allow that the space can reduce to something nontrivial of the type $N^O [\cdot]$. Observe that although the tetrad (4.10) with $\rho$ and $Q$ from (4.22) produces some "hellish" connections $\Gamma^a_{\beta \gamma}$, there exists an $SL(2, \mathbb{C})$ gauge in which these objects do vanish.

Now, as the next simple example we consider the function

$$
\Theta = - \frac{2}{\alpha^2} \frac{\beta}{\alpha - 1} x^y y^z, \quad \Theta = - \frac{2}{\alpha^2} \frac{\beta}{\alpha - 1} x^y y^z,
$$

which fulfills (3.10) for all values of constants $\alpha$ and $\beta$.

From (4.18) it follows that we have

$$
C^{(2)} = \partial \partial (1)^{a+2} (x^{a+2} y^{a-2} z),
$$

On the other hand, if $K^2 = 0$, then the equation for the $P$-spinor $K^A$, $C_{ABC} K^A K^K K^L = 0$ amounts to

$$
\frac{1}{2} \partial \partial \left( \frac{4}{a - 1} \right) \left( K^A / K^K \right)^{a-2} = 0.
$$

Substituting here from (4.25) we find that this equation is equivalent to

$$
\partial \partial (x K^A - y K^B y^{a-2} y^{a-2}) = 0,
$$

or explicitly

$$
(x K^A - y K^B) [\{ (\alpha + 2) (\alpha + 1) (x K^A) + (\alpha - 2) (\alpha - 3) (y K^B) \}
$$

$$
- 2 (\alpha + 1) (\alpha - 2) (x K^A (y K^B)) = 0.
$$

It follows that, irrespective of the value of $\alpha$, the spinor $K^A = (y, x)$ is always (at least) a double $P$-spinor, and that the factor with quadratic form in (4.27) has vanishing discriminant if and only if $\alpha = - 1$ or $\alpha = 2$. Therefore, $\partial \partial (x K^A - y K^B y^{a-2} y^{a-2}) = 0$ with $\alpha \neq - 1, 2$ it is of the type $[2 - 2] \otimes [-]$ and with $\alpha = - 1, 2$ it is of the type $[2 - 1 - 1] \times [-]$.

Therefore, we have succeeded in this section of producing explicitly examples of heavenly metric of all possible algebraically degenerate types.

Of course, this work and its results are to be considered as a technical step forward within the general goal toward which relativity is striving in recent years, i.e., to produce general techniques which would lead to general solutions of the Einstein equations on a real manifold. The complex solutions and complex geometry, although attracting interest at this moment (see, e.g., Refs. 8–10, ought to be considered only as an intermediate step.

ACKNOWLEDGMENT

I would like to express my gratitude to Dr. J.D. Finley for illuminating discussions and his active participation in deriving the results outlined in Sec. 2. A discussion with Dr. E.T. Newman and Dr. R. Penrose during the "Riddle of Gravity" symposium at Syracuse is also appreciated.

On leave of absence from University of Warsaw, Warsaw, Poland.


The null tetrad formalism used throughout this text, follows notation and convention of the real null tetrad formalism used.
in G. Debney, R. Kerr, and A. Schild, J. Math. Phys. 10, 1842 (1969), abandoning, however, of course the assumption that \((e^4)^*=e^4\) and all its implications.


The ordered symbol, however, describes the type of \(V_4\) as referred to the specific orientation of the tetrad considered given, modulo \(\gamma\) transformation only. Any \textit{improper} tetrad transformation \(e'^e = 4'\), with \(\text{det}(\theta)' = -1\), maps type \([A] \otimes [B]\) into \([B] \otimes [A]\); for that reason, if one thinks about the type of \(V_4\) irrespective of the orientation for the tetrad, we have to interpret \([A] \otimes [B]\) rather as a commutative tensor product.

\(^5\)That is, an \textit{oriented} complex \(V_4\) as per Ref. 5.

\(^6\)Of course, the improper tetrad transformation, e.g., \(e^1 \leftarrow e^2, e^2 \leftarrow e^1, e^3 \leftarrow e^4, e^4 \leftarrow e^3\), would change "strong heaven" into "strong hell" with \(\Gamma_{AB} = 0\).

