All nontwisting N's with cosmological constant

A. García Díaz and J. F. Plebański a)

Centro de Investigación y de Estudios Avanzados del I.P.N., Departamento de Física, Apdo. Postal 14-740, México 14, D.F., México

(Received 10 December 1980; accepted for publication 1 May 1981)

The set of all essentially different nontwisting type-N solutions with cosmological constant is presented in such a tetrad gauge, and coordinatization that the metric depends linearly on an arbitrary structural function. The special branches of the solutions of this type are shown to amount to the contractions of the most general branch.

PACS numbers: 04.20.Jb

I. INTRODUCTION

The aim of this work is to present all nontwisting N-type solutions with cosmological constant in such a representation that the metric depends linearly on the arbitrary structural complex function and to establish the interrelations among the different branches.

We employ the null tetrad formalism. Working with the signature + − 2, the metric is given by

\[ g = 2e^1 \otimes e^2 + 2e^3 \otimes e^4, \]

(1.1)

where \( e^2, e^3, e^4 = \tilde{e}^2, \tilde{e}^3, \tilde{e}^4 \). The first Cartan structure equations

\[ de^a = e^b \wedge \Gamma^a \]

(1.2)

define the connection 1-forms \( \Gamma_{ab} = \Gamma_{ba} \). Whenever the direction of \( e^2 \) is oriented along the quadruple Debever-Penrose vector, the second structure equations, into which the Einstein equations \( \left( \dot{g}_{\mu\nu}, \lambda \right) = \left( \lambda \right) \) are built in, can be written, for the N-type fields, as

\[ \dot{g}_{42} + \Gamma_{42} \wedge \dot{g}_{12} + \Gamma_{12} \wedge \dot{g}_{34} = \frac{\lambda}{2} e^3 \wedge e^1, \]

(1.3)

\[ \dot{g}_{12} + \Gamma_{12} + 2\dot{g}_{42} \wedge \dot{g}_{31} = -\frac{\lambda}{2} \left( e^1 \wedge e^2 + e^3 \wedge e^4 \right), \]

\[ \dot{g}_{31} + \left( \dot{g}_{12} + \dot{g}_{34} \right) \wedge \dot{g}_{31} = \frac{\lambda}{2} e^4 \wedge e^2 + \frac{1}{2} C^{(1)} e^3 \wedge e^1, \]

where \( C^{(1)} \) is the only nonvanishing conformal curvature coefficient.

In general, when \( e^3 \) is a distinguished direction, there arises a natural subclassification of the Riemannian structures invariant under \( \sigma \) and \( \gamma \) gauges,

\[ \sigma = \omega + i \phi: \quad e^{1} = e^{1} e^{-i \phi}, \quad e^{2} = e^{-i \phi} e^{3}, \quad e^{3} = e^{3} e^{1} e^{-i \phi}, \quad e^{4} = e^{-i \phi} e^{4}, \]

(1.4)

\[ \gamma: \quad e^{1} = e^{1} + \tilde{e}^{2}, \quad e^{2} = e^{2} + \tilde{e}^{3}, \quad e^{3} = e^{3} - \tilde{e}^{2}, \quad e^{4} = e^{4}. \]

which maintain the direction \( e^3 \) unchanged. These gauges induce on the connections the transformations

\[ \Gamma'_{42} = e^\phi \Gamma_{42}, \quad \Gamma'_{12} = \Gamma_{12} + \Gamma_{34} + d\sigma, \]

\[ \Gamma'_{31} = e^{-\phi} \Gamma_{31}, \]

(1.5)

and

\[ \Gamma'_{42} = \Gamma_{42}, \quad \Gamma'_{12} + \Gamma'_{34} = \Gamma_{12} + \Gamma_{34} + 2\psi \Gamma_{42}, \]

where \( \psi = \frac{\partial}{\partial \phi} \). This subclassification is:

1. First, there are structures with \( e^3 \) twisting (T) and nontwisting (NT). The twist of \( e^3 \) is defined as the 3-form

\[ T = e^3 \wedge e^1; \]

hence

\[ (T): \quad e^3 \wedge e^1 \neq 0, \]

\[ (NT): \quad e^3 \wedge e^1 = 0, \]

these conditions being manifestly invariant under (1.4)–(1.6).

Next, we consider the gauge invariant relative properties of \( \Gamma_{42} \) and \( e^1 \) under (1.4)–(1.6). One finds the following list of disjoint possibilities:

I: \( \Gamma_{42} = 0 \)

II: \( \Gamma_{42} \neq 0, \quad \Gamma_{42} \wedge \bar{\Gamma}_{42} = 0 \)

\[ \left\{ \begin{array}{l} \Pi_{2}: \quad e^3 \wedge \Gamma_{42} = 0, \\ \Pi_{3}: \quad e^3 \wedge \bar{\Gamma}_{42} = 0, \end{array} \right. \]

(1.9)

III: \( \Gamma_{42} \wedge \bar{\Gamma}_{42} \neq 0 \)

\[ \left\{ \begin{array}{l} \Pi_{4} \quad e^3 \wedge \Gamma_{42} \wedge \bar{\Gamma}_{42} = 0, \\ \Pi_{5}: \quad e^3 \wedge \bar{\Gamma}_{42} = 0, \end{array} \right. \]

(1.10)

The contravariant metrical density is given by

\[ g[\phi, \phi] = e^\phi \sigma_{\nu} \phi_{\mu}, \]

\[ = -2 \phi_{\rho} \phi_{\xi} - 2 \phi_{1} \left[ \phi_{\rho} - (\phi_{1} + \phi_{2}) \phi_{\rho} \right], \]

(2.2)

where \( e = \det(e_{\mu}^a) \) and \( \phi \) is an arbitrary function.

Because these \( R \) waves are prototypic for the other branches of the N problem studied, we would like to add some comments concerning their properties. Notice that the
tangent tetrad and the contravariant density are linear in the free structural function \(f(\xi, t)\). The dependence of \(f(\xi, t)\) on the variable \(t\) is completely free. This variable defines a set of characteristic surfaces \(t = \text{const}\) of the metric satisfying the eikonal equation \(g(t, t) = 0\). With the dependence of \(f\) on \(t\) entirely free, the amplitude of the curvature \(C(3)\) is determined by an analytic function chosen arbitrarily along each of the characteristic surfaces \(t = \text{const}\). These properties are common to all the nontwisting \(N\)-type solutions, as we shall see later.

III. ALL SOLUTIONS OF CASE II

One can show that nontrivial solutions may exist only in the subbranch \(II_a\). The direction \(e^2\) is then geodesic, shearless, and free of complex expansion.

These solutions new to our knowledge, which we denote by \(K (\lambda)\) waves, are the generalization of the Kundt solution \(K\) and are given via:

(i) the tangent null tetrad:
\[
\begin{align*}
\partial_1 &= \cosh x \partial_x, & \partial_2 &= \cosh x \partial_x, & \partial_3 &= \cosh x \partial_x, & \partial_4 &= \cosh x \partial_x, \\
\frac{\sinh x}{\sqrt{\mu}} \partial_3 &= \partial_1 + r \cosh x (\partial_2 + \partial_4) \\
- \cosh x (\partial_2 + \partial_4 - 2(\sqrt{\mu}) \tan hy (f + \tilde{f}) (\sqrt{\mu})^2 \tan hy ) \partial_3,
\end{align*}
\]

where \(\mu = \lambda / 6, x = (\sqrt{\mu}) \xi + \tilde{\xi}\);

(ii) the conformal curvature:
\[
C(3) = -2(\lambda / 6) (\cosh x / \sinh x) |\partial_2 \partial_4 - \lambda | \partial_x |
\]
and

(iii) the connection form \(\Gamma_{a2}\):
\[
\Gamma_{a2} = (\sqrt{\mu / \sinh x}) e^1. \tag{3.3}
\]

The metrical density is then
\[
- g[\phi \phi] = \frac{2(\sqrt{\mu}) \sinh x / \cosh x}{x (d_2 \phi_2 + (\sqrt{\mu}) \sinh x) \phi_1 + r \cosh x (\phi_2 + \phi_4) - \cosh x (\partial_2 + \partial_4 - 2(\sqrt{\mu}) \tan hy (f + \tilde{f}) (\sqrt{\mu})^2 \tan hy ) \phi_3 - (\sqrt{\mu})^2 \tan hy ) \phi_1]. \tag{3.4}
\]

Here \(\lambda\) is assumed to be positive; for \(\lambda < 0\), the hyperbolic functions are to be replaced by the corresponding trigonometric ones.

As in the case of the Robinson waves, the tetrad and the metrical density are linear in the free structural function \(f(\xi, t)\). The variable \(t\) determines the characteristics of the metric, and the analytic function which determines the amplitude of the curvature can be arbitrarily chosen along each characteristic surface.

IV. ALL NONTWISTING SOLUTIONS OF CASE III

Within this class there exist solutions of the type \(N\) only in the subbranch \(II_a\). We restrict ourselves here to study the diverging but free of rotation \(N\)-solutions. These were investigated by Leroy, who obtained basic results following the theory of Robinson and Trautman. However, in his treatment, the solution was given modulo some equations. Also, his choice for the tetrad did not exhibit the linear dependence on an arbitrary structural function. We have succeeded, we believe, in deriving a much more satisfactory description of the solutions of this type, which depend linearly on an arbitrary analytic function.

Exploiting the freedom of gauges, we succeeded to integrate equations (1.2) and (1.3) determining an "optimal" chart of coordinates, \(\xi, \tilde{\xi}, r, t\), and the corresponding tetrad gauge in which all nontwisting \(N\) waves with \(\lambda\) are given by
\[
e^1 = rd_2 + (\psi_2 - r') dt, \quad e^2 = (\tilde{\xi}',
\]
\[
e^3 = \psi dt,
\]
\[
e^4 = dr - \psi_2 + [(r_2 + \tilde{f}_2) + \lambda r^2 \psi] dt, \tag{4.1}
\]
where \(f = f(\xi, t)\) is an arbitrary complex function depending on \(\xi\) and \(t\) only.

The tangent tetrad associated with (4.1) is
\[
\begin{align*}
\partial_1 &= \partial_x, & \partial_2 &= \partial_x, & \partial_3 &= \partial_x, & \partial_4 &= \partial_x, \\
\psi_2 &= \partial_x - (1/r_f) [r (\psi_2 - r \tilde{f}_2) + \psi_2] - |r - \psi_2 + 1/4 (f_2 + \tilde{f}_2) + \lambda \psi r^2 | \partial_x.
\end{align*}
\]

By using appropriately the remaining coordinate freedom, the structural function \(\psi\), without any loss of generality, can be always brought to the form
\[
\psi = 1 + \varepsilon_2 \tilde{\xi}, \tag{4.3}
\]
where the discrete parameter \(\varepsilon\) takes the values 1, 0, -1. Optionally, for \(\varepsilon = -1\), we represent \(\psi\) as
\[
\psi = (1 + \sqrt{a}) \sinh x, \quad x = (\sqrt{a}) \xi + \tilde{\xi}, \quad a = \text{const}. \tag{4.4}
\]

[Here \(a\) is assumed to be positive; for a negative \(\psi\)
\[-(1 + \sqrt{a}) \sinh x\]. The connections accompanying our tetrad are
\[
\begin{align*}
\Gamma_{a2} &= - (1/r) e^1 + (1/r \sqrt{a}) e^3, \\
\Gamma_{a1} + \Gamma_{a3} &= (\sqrt{a} \lambda + \psi - r^3) e^3, \\
\Gamma_{a2} &= (1/r \sqrt{a}) e^1 + \lambda r e^3
\end{align*}
\]
+ \(1/r \sqrt{a}) (\psi_2 e^1 + \lambda r e^3 + \lambda r e^3). \tag{4.5}
\]

In particular, for \(\psi\) given by (4.3) the connections above take the simple form
\[
\begin{align*}
\Gamma_{a2} &= - (1/r) e^1 + (1/r \sqrt{a}) e^3, \\
\Gamma_{a1} + \Gamma_{a3} &= (\sqrt{a} \lambda + \psi - r^3) e^3, \\
\Gamma_{a2} &= (1/r \sqrt{a}) e^1 + \lambda r e^3 + (1/2 \psi) (f_2 + \lambda r e^3). \tag{4.6}
\end{align*}
\]

The conformal curvatures, corresponding to the \(\psi\)'s given by (4.3) and (4.4), are
\[
C(3) = - f^2 \sqrt{1 + \varepsilon_2 \tilde{\xi}} \ne 0 \tag{4.7}
\]
and
\[
C(3) = - (1/r) ([\sqrt{a} \sinh x] [\partial_2 \partial_4 - 4a] f_2) \ne 0, \tag{4.8}
\]
respectively.

Note that the tetrad, from (4.1), is linear in the structural function \(f(\xi, t)\); note also that the variable \(r\) defines the set of

A. Garcia Dáaz and J. F. Plebanski 2656
characteristic surfaces of the metric, \( t = \text{const.} \)

In what follows we shall denote this class of solutions by \( \text{NT}(\lambda, Z,e) \), where \( \lambda \) stands for the cosmological constant, \( Z \) represents the complex expansion \(-\Gamma_{22} \), and the parameter \( e \) takes the values \( 1, 0, -1 \), depending upon whether the "source" lines of the gravitational waves are respectively timelike, null or spacelike. The complex expansion \( Z \) in the studied problem is real and equal to \( 1/r \); therefore, it represents the divergence of the congruence \( e \).

All vacuum nontwisting solutions of the type \( N \), denoted as \( \text{NT}(0,Z,e) \), are obtained from the \( \text{NT}(\lambda, Z,e) \) by simply equating \( \lambda \) to zero in the expressions above. These solutions were obtained in Ref. 8; see also Refs. 9 and 10. Nevertheless, the tetrad gauge used there does not exhibit a linear dependence on an arbitrary structural function. The simple form representations of these solutions, given in this text, having a linear dependence on an arbitrary structural function. The simple form of these solutions, given in this text, having a linear dependence on the structural function, facilitates, among other things, the limiting transitions to the subcases of Kundt and Robinson which depend linearly on the structural function.

V. CONTRACTIONS

The purpose of this section is to show that the sub-

\[
\begin{align*}
\rho_1 &= \rho_2 = \rho_4 = \partial_\xi - \sqrt{2} \sinh \rho \partial_\eta, \quad \rho_3 = (\sqrt{2} \sinh \rho) \partial_\eta, \\
\rho_5 &= \frac{1}{\sqrt{a}} \cosh \rho \partial_\eta - \frac{1}{r} (\cosh \rho - \sqrt{2} \sinh \rho \partial_\xi - \frac{1}{r} (\cosh \rho - \sqrt{2} \sinh \rho \partial_\xi) \partial_\eta \\
&\quad - \left( -(\sqrt{a}) \sinh \rho + \frac{r}{2} (\xi + \xi) + \frac{\lambda^2}{6} \sinh \rho + \frac{v^2 (\sqrt{a}) \rho^2 \sinh \rho}{\rho} \right) \partial_\xi, \\
C^\text{(II)} &= -\frac{\cosh \rho}{r} \frac{\sqrt{a}}{\sinh \rho} (\partial_\xi \partial_\eta - 4a) f_\xi, \\
x &= (\sqrt{a}) (\xi + \xi).
\end{align*}
\]

We change now the coordinates and other quantities which appear above according to

\[
\begin{align*}
t &= e^{-t'} \xi, \quad \xi = e^\xi, \quad \xi = e^\xi, \\
r &= \frac{1}{\cosh \rho} \left( \frac{r'}{\cosh \rho} + e^{-1} \right), \\
v &= e, \quad a = e^{\lambda e^{-2}}, \quad f = e(2e^2 f' + 1),
\end{align*}
\]

where the function \( f'(\xi', t') \) is assumed to be independent of the parameter \( e \). Making \( e \) tend to zero, we obtain precisely the \( K(\lambda) \) solutions given by (3.1)-(3.2).

We now consider the pertinent contractions of the \( \text{NT}(0,Z,e) \) solutions. The Kundt metrics happen to be the contraction of the \( \text{NT}(0,Z, -1) \) waves. Indeed, setting \( e = -1 \) in (4.3) and \( \lambda = 0 \) in (4.2) and transforming the coordinates and the structural function according to

\[
\begin{align*}
t &= e^{-t'} \xi, \quad \xi = e^\xi, \quad \xi = e^\xi, \\
r &= \frac{1}{\cosh \rho} \left( \frac{r'}{\cosh \rho} + e^{-1} \right), \\
v &= e, \quad a = e^{\lambda e^{-2}}, \quad f = e(2e^2 f' + 1),
\end{align*}
\]

by letting \( e \) go to zero, we arrive at a slightly modified version of the Kundt waves

\[
\begin{align*}
\partial_1 &= \partial_\xi, \quad \partial_2 = \partial_\xi, \quad \partial_4 = \partial_\eta, \\
\partial_3 &= \partial_\xi + r (\partial_\xi + \xi) + (r - f - f) \partial_\eta,
\end{align*}
\]

branches \( K(\lambda) \), \( K, R \) can be derived from the \( \text{NT}(\lambda, Z,e) \) via corresponding limiting transitions.

From expressions (3.1)-(3.2), by letting \( \lambda \) go to zero and at the same time changing \( f_\xi \to f \), we readily obtain the Kundt waves \( K \)

\[
\begin{align*}
\partial_1 &= \partial_\xi, \quad \partial_2 = \partial_\xi, \\
\partial_3 &= \partial_\xi + r (\partial_\xi + \xi) + (r - f - f) \partial_\eta,
\end{align*}
\]

VI. CONCLUSION

We have found that all geometrically different nontwisting N-type solutions with \( \lambda \) are interrelated, by contractions, according to the diagram given below.

A. García Díaz and J. F. Plebański
Therefore, the set of $N\{i, Z, \varepsilon\}$ solutions completely ex-
hasts all $N$-type nontwisting solutions.

1. J. F. Plebański; *Spinors, Tetradic and Forms*, a monograph of Centro de
3. I. Robinson, Lecture at King's College, London (1956). Report to the
Rayaumont Conference (1959).
10. D. Kramer, H. Stephani, M. Mac Callum, and E. Herlt; *Exact Solutions of
261.