Signals and discontinuities in general relativistic nonlinear electrodynamics

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A theory of nonlinear electrodynamics in an arbitrary curved space–time is developed from the fundamental action functional for a charged perfect fluid. The equations for small perturbations on a fixed nonlinear background are then the initial point for a comprehensive study of the characteristic surfaces. The essential distinctions between linear and nonlinear electrodynamic interactions under the influence of gravitation are exhibited. Discontinuities in the first derivatives of small perturbations are encountered (1) which may be of general algebraic types for both the electrodynamic and gravitational fields and (2) which may have spacelike propagation. A specific set of constraints which would permit the propagation of these extraordinary radiative fronts is presented. If the physical organization of a particular problem is presumed to be sufficiently sensitive to the nonlinear nature of the dynamical interactions, then the application of traditional causal concepts may be unreliable when intuition derived from Maxwellian electrodynamics with noninteracting photons is anticipated to provide event horizons.

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1. INTRODUCTION

This paper has two basic objectives. The first is to present a compact covariant exposition of nonlinear electrodynamics (NLE) in a curved space–time (i.e., the dynamics of the Einstein–Born–Infeld equations with a general structure function), investigating the algebraic aspects of the theory by spinorial techniques. The second is to offer a comprehensive study of the characteristic surfaces of such a system, demonstrating the essential distinctions between the linear and nonlinear electrodynamic cases.

The actual content of the work is a generalization of Ref. 1, drawing extensively from the calculations presented in Ref. 2. The paper continues the sequential study of NLE with its vast literature1–13 initiated by Born and Infeld. Because of the similarities between Refs. 1 and 13, the results may also be thought of as an extension of those of Boillant. Recently these ideas of NLE have been found of interest even in supersymmetric theories.14

There are two physical approaches to the theory. The first is to seek, via nonlinearity, a structurally self-consistent classical electrodynamics with a finite point charge inertia which is free of the conceptual difficulties related to the divergences which plague the linear theory. Within this approach, with a sensible structure function of invariants, NLE is capable of fulfilling all common sense requirements (like, e.g., the correct transformation properties for finite conserved quantities of a point charge) in contrast to other variants, including those with extended sources, higher derivatives, form factors, etc., each of which retains some incompatibility. Unfortunately, the shape of the structure function remains remarkably arbitrary, which handicaps specific physical predictions from the theory. This is the main reason for the limited interest in these matters in general theoretical physics. The second approach regards NLE as a variety of phenomenological quantum electrodynamics (QED) in the limit of high occupation numbers. The structure function is selected such that the QED predictions concerning the scattering of light by light are reproduced classically by the NLE. Of particular interest is the structure function of Schwinger,15 which accounts for an infinite ladder of quantum processes. Our motivations are closer to the second of these options.

If electromagnetic fields in curved space–times are critical to the early evolution of the universe or for the dynamics of collapsing objects near singularity limits, then it is natural to expect the corresponding QED dependent processes will affect the physics. Hence, NLE provides a simple tool for evaluating possible implications closer to physical reality. It is rather naive to anticipate that classical Maxwellian electrodynamics with noninteracting photons can reliably represent such extremal conditions. Consequently, it is interesting to study the nature of causal signals within Einstein–Born–Infeld dynamics. The original Einstein construction of causal cones is related to light or, more specifically, to the surfaces along which discontinuities of the first derivatives of the electromagnetic field (characteristic surfaces) of linear electrodynamics are propagated. Light rays are then understood as bicharacteristic lines. This identification forms the foundation of special relativity, and without alteration it is extrapolated into the realm of general relativity. In fact, in general relativity, when a linear Maxwellian field is present, its characteristics do coincide with the Einsteinian γμν cone. This, however, will not be the case when NLE governs electromagnetic phenomena. This result permits the reinterpretation of the standard characteristic of general relativity, the eiconal equation 2 gμνSμSν = 0, which is a major objective of this paper. It is not unreasonable to assume that the introduction of other forms of nonlinear interactions (e.g., fluid dynamic or Yang–Mills fields) would produce analogous results.

Formally, the paper is organized such that Sec. 2 contains the generation of the dynamic equations for a perfect charged fluid, containing a nonlinear electromagnetic interaction within a curved space–time, by variations of the fundamental action. The structure equations which contain the distinction between the linear and nonlinear electrodynamics...
cases are introduced. Section 3 provides the conversion of the equations into their spinor counterparts and reviews the algebraic properties of the electromagnetic field tensors critical to the evaluation of the characteristic surfaces. There is also a brief discussion of how the NLE presented can be interpreted as analogous to more conventional electrodynamics within a medium. The entire self-consistent system is linearized by standard first perturbation techniques and the “jump” expressions satisfied by the discontinuities in the first derivatives of the field variables are given in Sec. 4. Section 5 demonstrates how the system reduces identically to the standard results in the linear electrodynamic limits. In particular, the algebraic investigation indicates that in the linear limit only null characteristic surfaces permit the existence of nontrivial jumps. Section 6 classifies in detail the necessary conditions for the jump expressions for the many possible characteristics determined by the structure function in the NLE case. Dependent upon the nonlinearity, the results indicate that the discontinuities of both the gravitational and electromagnetic fields may be of algebraically general types with respect of the Einsteinian metric. The Einsteinian local null cone is physically identified only with the propagation of pure gravitational radiative fronts, and there exist nonlinearly interacting massless fields (including the electromagnetic) whose propagation may be associated with causal cones both interior and exterior to the Einsteinian null cone. Assuming there exist physically reasonable nonlinear structure functions, satisfying a rather simple set of conditions, these different cones which allow causal influence certainly do no coincide. The distinguished rate of propagation of discontinuities in conformal curvature alters the local measurement of time and thereby changes our traditional view of the propagation of linear electromagnetic fronts as the only provider of a causal horizon for an event. The metric properties for each form of characteristic surface are then cataloged in Sec. 7, and finally Sec. 8 concludes by displaying the algebraic types of jumps in tabular form.

2. THE DYNAMIC EQUATIONS

The relevant dynamic equations are evaluated by vari­ations of the action functional

$$\mathcal{S} = \int d^4x \sqrt{-g} \left( \mathcal{L}_F + \mathcal{L}_E + \mathcal{L}_I + \mathcal{L}_G \right),$$

(2.1)

where the infinitesimal four-volume element $d^4x \sqrt{-g}$ is invariant under nonsingular coordinate transformations within a Riemannian background space-time $V_4$ represented by the metric $g_{\mu\nu}$ with signature $(+++)$ and determinant $g$. The associated Lagrangians correspond to the fluid $\mathcal{L}_F$, the nonlinear electromagnetic field $\mathcal{L}_E$, the interaction between the fluid and field $\mathcal{L}_I$, and the background gravitational field $\mathcal{L}_G$, respectively. The line integrals are understood to be taken along the world line of fluid particles between the points of intersection of two spacelike three-surfaces bounding the domain $\Omega$ of variation.

Assuming there exists a known equation of state such that the energy density is given by $\epsilon = \epsilon(n,s)$, where $n$ denotes the particle density and $s$ the entropy density of a fluid point, the fluid flow may be characterized by (1) an Eulerian velocity $u^\mu(x^n)$ defined for every event $P(x^n)$ situated on its world trajectory, (2) the specific volume $V = 1/n$, and (3) the phenomenologic temperature $T$ and pressure $p$ encountered. These hydrodynamic variables are measured with respect to a local rest frame. The velocity is normalized such that $u^\mu u_\mu = -1$. Since the fluid is isotropic and frictionless, entropy is conserved. Therefore,

a) $u^\mu s_\mu = 0$

and

b) $(nu^\mu)_{,\mu} = 0$.

(2.2)

The first and second laws of thermodynamics,

$$d\epsilon = nTds + [(\epsilon + p)/n]dn,$$

(2.3)

are postulated. Associated with such a fluid is the Lagrangian density

$$\mathcal{L}_F = -\epsilon(n,s).$$

(2.4)

The nonlinear electrodynamic field is represented by two skew field tensors $f^{\mu \nu}$ and $P_{\mu \nu}$, which are interrelated through a single relation designated as the "structure" equation. The existence of a potential $A$ satisfying the Faraday field equation

$$f_{\mu \nu} = A_{\mu \nu} - A_{\nu \mu}$$

(2.5)

is assumed, yielding the electrodynamic Lagrangian

$$\mathcal{L}_E = -(1/4\pi)[f_{\mu \nu} P^{\mu \nu} - H(P,Q)].$$

(2.6)

The "structure function" $H = H(P,Q)$, whose arguments are invariant (scalar) and pseudoinvariant, respectively, is a real Hamiltonian whose functional form is intentionally left unspecified. For linear electrodynamics $H = P$, but for nonlinear electrodynamics $H$ must only conform to a couple of general conditions. One expects for the weak field [$P,Q$ small] limit that nonlinear effects will become negligible:

$$\text{correspondence} \rightarrow H(P,Q) = P + O(P^2,Q^2) \Rightarrow H(0,0) = 1.$$  

(2.8)

If parity is conserved, then under coordinate transformations with negative Jacobian, where $Q$ transforms into $-Q$, $H$ must remain invariant

$$\text{parity conservation} \rightarrow H(P,Q) = H(P,-Q),$$

(2.9)

which is equivalent to $H = H(\bar{P},\bar{Q})$. Condition (2.9) is less essential than (2.8), since one could consider systems with parity violating weak interactions in a quantized theory. However, in this work, both restrictions (2.8) and (2.9) are presumed satisfied. Additionally, the conservation of charge density $\rho$,

$$p_{\mu} = 0,$$

(2.10)

is required.

The interaction between the fluid and the nonlinear electromagnetic field is provided by the minimal coupling Lagrangian density

$$\mathcal{L}_I = A_{\mu} J^{\mu},$$

(2.11)

where $J^{\mu} = \rho u^\mu$ denotes the electric current density. Notice
that the expressions determining the inertia of the fluid (2.4) and the coupling with the electromagnetic field (2.11) are identical to those of linear electrodynamics.

The gravitation field equations are deduced from the conventional Lagrangian density of Einsteinian gravity,

\[ \mathcal{L}_g = \{1/16\pi\}(R + 2\lambda), \]  

(2.12)

written in gravitational units \((G = \text{gravitational constant} = 1 = \text{velocity of light} = c), \) with scalar curvature \(R\) and cosmological constant \(\lambda.\) Restricting this work to special relativity \(\text{(in general coordinates)}\) by submitting the metric \(g_{\mu\nu}\) to the condition of vanishing Riemann tensor \((R^{\mu\nu}_{\rho\sigma} = 0)\) yields the results previously given in Ref. 1.

The dynamic equations are derived from the overall action \(\mathcal{A}\) of (2.1) accounting for (2.4), (2.6), (2.11), and (2.12) by executing the variations and extremalizing with respect to \(G\) the Lagrangian trajectories \((\delta \mathcal{L}_g),\) (2) the potential \(\{\delta A_{\mu}\},\) (3) the field \(\mathcal{L}_{\mu\nu}\) regarded as Lagrange multiplier \(\{\delta P_{\mu\nu}\},\) (4) the metric \(\{\delta g_{\mu\nu}\},\) and (5) the connections \(\Gamma^{\mu}_{\alpha\beta}\) considered as independent of the metric \(\{\delta \Gamma^{\mu}_{\alpha\beta}\}\) \(\text{(according to the principle of Palatini).}\) The process must be consistent with the subsidiary conservation conditions (2.2) and (2.10), and constrained by null variations on the boundary \(\partial \Omega\) of the region \(\Omega.\) Performing the extremalizations, the following equations of evolution of the perfect fluid system with nonlinear electrodynamical and gravitational fields are evaluated. The details are found in Ref. 2.

\[ \delta \mathcal{A} / \delta \Gamma^{\mu}_{\alpha\beta} = 0 \to \delta \mathcal{R}^{\mu}_{\nu\rho} - \delta^{\alpha}_{\nu} \delta^{\beta}_{\rho} = 0 \to g_{\mu\nu} = 0, \]  

(2.13)

where \(\delta \mathcal{R}^{\mu}_{\nu\rho} = (g)^{1/2} \delta g_{\mu\nu}.\) Consequently, the appropriate connection for the metric is that of Levi-Civita \((\Gamma^{\mu}_{\nu\rho})\). The equations of motion (Lorentz equations) for the charged fluid are

\[ \delta \mathcal{A} / \delta \mathcal{E} = 0 \to \delta T^{\mu}_{\nu} + \delta^{\nu}_{\alpha} \delta^{\mu}_{\rho} = 0 \to g_{\mu\nu} = 0, \]  

(2.14)

where \(\delta T^{\mu}_{\nu} = \delta \mathcal{E}^{\nu}_{\alpha} + u^{\alpha} \delta \mathcal{E}^{\nu}_{\alpha} = 0 \) and the energy-momentum tensor is defined by \(T^{\mu}_{\nu} = \rho g^{\mu\nu} + (e + p)u^{\mu}u^{\nu}.\) The electromagnetic field (Maxwell's) equations are

\[ \delta \mathcal{A} / \delta A_{\mu} = 0 \to P^{\mu}_{\nu} = 4\pi J^{\mu}. \]  

(2.15)

These equations are complemented by the Faraday equations \(\text{(the existence of the potential)}, \) which express the necessary and sufficient conditions that \(f^{\mu}_{\nu}\), be a curl,

\[ f^{\mu}_{\nu} = \partial f^{\mu\nu\lambda} / \partial x^{\lambda} = 0 \Rightarrow f^{\mu}_{\nu} = A_{\mu\nu} \]  

(2.16)

The structure equations (material equations)

\[ \delta \mathcal{A} / \delta P^{\mu}_{\nu} = 0 \to f^{\mu}_{\nu} = 2\partial H / \partial P^{\mu}_{\nu} = H_{\rho} P^{\rho\mu} + H_{\delta} f^{\delta}_{\mu}, \]  

(2.17)

are the counterparts of the Lorentz material equations in the classical electrodynamics of polarized media. Equations (2.15)–(2.17) give the physical interpretation and the evolution of the nonlinear electromagnetic field whose energy-momentum tensor is defined by

\[ 4\pi E^{\mu\nu} = -f^{\mu\nu} + g^{\mu\nu}L \text{ with } L = -4\pi \mathcal{E}. \]  

(2.18)

For a linear field, \(H = P - f^{\mu}_{\nu} = P_{\mu\nu}.\) Varying the metric, one obtains the gravitation equations (Einstein equations

\[ \delta \mathcal{A} / \delta g_{\mu\nu} = 0 \to G^{\mu\nu} = 8\pi(T^{\mu\nu} + E^{\mu\nu}) + \lambda g^{\mu\nu}, \]  

(2.19)

where \(G^{\mu\nu} = R^{\mu\nu} - g^{\mu\nu}R\) is the Einstein tensor. The expression (2.19) and the geometric structure equations (Bianchi identities)

\[ R^{\alpha\beta}_{\gamma\delta} = 0 \]  

(2.20a)

and

\[ G^{\mu\nu}_{\cdot\cdot} = 0 \]  

(2.20b)

govern the evolution of space-time and the motion of material contained within it. This essentially completes the system of dynamical equations. The conservation constraints (2.2) and (2.10) have been inherently imposed. When this system is restricted to special relativity, where the Riemann tensor vanishes, the total energy-momentum tensor is still divergenceless as (2.20b) implies, but now as the result of the translational invariance of the action.

Since the essential distinction between linear and nonlinear electrodynamics resides in the electromagnetic structure equations (2.17), it is informative from the outset to inquire about their inversion. When these equations can be inverted one may algebraically express \(P_{\mu\nu}\) through \(f^{\mu}_{\nu},\) its dual, and the invariants

\[ F = \frac{1}{2} f^{\mu}_{\nu} f^{\nu}_{\mu} \text{ and } \tilde{G} = \frac{1}{2} f^{\mu}_{\nu} \tilde{f}^{\nu}_{\mu}, \]  

(2.21)

by the expression

\[ P_{\mu\nu} = 2 \partial L / \partial f^{\mu}_{\nu} = L_f f^{\mu}_{\nu} + L_d \tilde{f}^{\mu}_{\nu}. \]  

Hence \(L = (F, \tilde{G})\) is understood as a function of \(f^{\mu}_{\nu},\) depending on it through the arguments \(F\) and \(\tilde{G}.\) The relations among invariants can be represented by a complex equation in either of the forms

\[ F + \tilde{G} = (H + \tilde{H}) \]  

(2.23a)

or

\[ P + \tilde{Q} = (F_L + \tilde{L}) \]  

(2.23b)

Thus, if \(\text{if}[2.23]\) can be inverted \(\text{[i.e., } \partial [F, \tilde{G}] / \partial (P, \tilde{Q}) \neq 0\text{]}\) determining \(P = (F, \tilde{G})\) and \(\tilde{Q} = (F, \tilde{G}),\) then

\[ L = (F, \tilde{G}) = 2PH_L + 2\tilde{Q}H_{\tilde{Q}} - H. \]  

(2.24)

Moreover, (2.22) substituted into the definition of \(L\) \([2.6]\) and \([2.18]\) implies

\[ H(P, \tilde{Q}) = 2FL_F + 2\tilde{G}L_{\tilde{Q}} - L. \]  

(2.25)

The self-consistency of the theory is demonstrated by showing that the divergence of the total energy-momentum tensor, indicated in the special Bianchi identity (2.20b), vanishes. Sequentially, this is accomplished by applying the equations of Lorentz, Maxwell, and Faraday, the definition of the dual, the electromagnetic structure relation, the anti-symmetry property of \(P^{\mu\nu},\) and finally the definition of \(E^{\mu\nu}.\)

3. SPINOR FORM OF THE EQUATIONS

Since the classification of the characteristic surfaces is conveniently done in spinor formalism, the spinorial counterparts of the dynamic equations are now presented. Using the Hermitian Pauli matrices \(g^{\alpha}_{\beta},\) the self-dual spin tensor \(S^{\alpha}_{\beta\gamma},\) and its anti-self-dual complex conjugate \(S^{\nu}_{\alpha\beta},\) the spinorial images of all relevant objects may be defined in

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the usual manner. The spinor covariant derivative operator is \( \nabla_{\gamma ab} = g_{\gamma ab} \nabla_{\gamma} \). For the symbols corresponds identically to those used previously, the equations of motion (2.14), Maxwell (2.15), and Faraday (2.16), respectively, become

\[
u_{ab} \nabla_{\gamma} \Phi + \nabla_{\gamma} \phi + |e + p| \mu_{ss} \nabla_{ss} u_{ab} = 2\mathcal{J}^a_s + F_{ab} \mu_s^a, \tag{3.1}
\]

\[
\nabla_{s} \Phi + \nabla_{s} \phi = 4\pi J^a_s, \tag{3.2}
\]

\[
\nabla_{s} J^a_s - \nabla_{s} \phi = 0 \Rightarrow j_{ab} = -\nabla_{s} A_{bs} - \nabla_{s} A_{ab}, \tag{3.3}
\]

complemented by the conservation of charge (2.10)

\[
\nabla_{\gamma} \Phi_{ab} = 0. \tag{3.10}
\]

The electromagnetic field (material) equations (2.17) and their inverses (2.22) are most conveniently written in their equivalent spinor relations after the introduction of

\[
Z = P + \tilde{Q} = 4P^{ab} P_{ab} \quad \text{and} \quad W = \bar{F} + \tilde{G} = 4f^{ab} f_{ab} \tag{3.4}
\]

and the respective complex conjugates as the independent invariants. We obtain

\[
\text{with} \ f_{ab} = \frac{2H_j}{J} P_{ab} , \tag{3.5a}
\]

and

\[
\frac{1}{2} (H_j)_{ab} = \frac{1}{2} (L_j + L_{\tilde{j}}), \tag{3.5b}
\]

which are equivalent to

\[
f_{ab} = (H_j + H_{\tilde{j}}) P_{ab} , \tag{3.6a}
\]

and

\[
\frac{1}{2} P_{ab} = (H_j - H_{\tilde{j}}) f_{ab} . \tag{3.6b}
\]

Moreover, (3.6) implies

\[
(H_j + H_{\tilde{j}}) (L_j + L_{\tilde{j}}) = 1. \tag{3.7}
\]

For the symmetric tensors, the trace is extracted before contracting each index independently with \( j \) of the general Pauli matrices. The traceless energy-momentum tensor of the fluid is \( F^{\nu\nu} = T^{\nu\nu} - \frac{1}{4}g^{\nu\nu} T \). Hence it has a spinorial image denoted by

\[
F_{\gamma\nu\mu\rho} = g_{\gamma\nu\rho} f^{\nu\rho} g_{\rho\mu} , \tag{3.8}
\]

and therefore,

\[
F_{\gamma\nu\mu\rho} = (e + p) (\mu_{\nu\rho} u_{\mu} - \frac{1}{2} \mu_{\nu\rho} e_{\mu}) . \tag{3.9}
\]

Similarly, the traceless nonlinear electromagnetic field energy-momentum becomes

\[
\pi E_{\gamma\nu\mu\rho} = -H_j P_{ab} P_{cd} = -L f_{ab} f_{cd} . \tag{3.10}
\]

The image of the traceless Einstein tensor \( C^{\nu\mu} = G^{\nu\mu} + \frac{2}{3} R \) reduces the Einstein equations to the following relations:

\[
C_{\gamma\nu\mu\rho} = 8\pi (F_{\gamma\nu\mu\rho} + E_{\gamma\nu\mu\rho}), \tag{3.11a}
\]

\[
= 8\pi (e + p) (\mu_{\nu\rho} u_{\mu} - \frac{1}{2} \mu_{\nu\rho} e_{\mu}) - 8H_j P_{ab} P_{cd} + \nabla_{\gamma} R = -2\pi [3p + \frac{2}{4} (\nabla_{\gamma} P + \tilde{Q} H_{\tilde{j}})] . \tag{3.11b}
\]

Denoting the conventional conformal curvature by the totally symmetric and complex object

\[
C_{\gamma\nu\mu\rho} = \frac{1}{2} S^{\gamma\nu\mu\rho} S_{\gamma\nu\mu\rho} , \tag{3.12}
\]

the Bianchi identifies (2.20) yield

\[
\nabla_{\gamma} C_{\gamma\nu\mu\rho} + \nabla_{\nu} C_{\gamma\nu\mu\rho} \partial_{\gamma} = 0, \tag{3.13a}
\]

\[
\nabla_{\gamma} C_{\gamma\nu\mu\rho} + \nabla_{\nu} C_{\gamma\nu\mu\rho} \partial_{\gamma} = 0, \tag{3.13b}
\]

Particle conservation (2.2b) and velocity normalization take the forms \( \nabla_{\gamma} (au_{\gamma}) = 0 \) and \( u_{\gamma} u^\gamma = 2 \).

The original tensors are recovered from the spinorial images by applying the inversion relations originating from the duality properties of the spin tensor and its complex conjugate and the normalization of the Pauli matrices

\[
(\delta_{\nu} \mu - \frac{1}{2} \epsilon_{ab} g^{\nu\mu}) \text{ given by}
\]

\[
\frac{1}{2} s_{ab}^{\nu} s_{ab}^\mu = \delta_{a}^{\nu} \delta_{ab} + (i/\sqrt{-g}) \epsilon_{ab} \nu^5
\]

and its complex conjugate.

We next recall briefly some algebraic properties of the electromagnetic field tensors which are necessary for determining characteristic surfaces. A discussion of greater depth from both the mathematical and physical points of view may be found in Ref. 1 or 20. Considerations of gauge freedom can also be found there. The electromagnetic field tensor \( f_{\nu\mu} \) is a real skew symmetric second rank tensor which is called simple if \( G = 0 \), null if \( F + G = 0 \), and algebraically general if \( F + G \neq 0 \). If \( f_{\nu\mu} \), is simple, then (and only then) there exist real \( a_\mu \) and \( b_\nu \) such that

\[
(a_\mu b_\nu = a_\nu b_\mu - a_\mu b_\nu .
\]

The spinorial image \( f_{ab} = f_{ab} \), always has complex factorization

\[
f_{ab} = a\alpha_1 \beta_\mu . \tag{3.15}
\]

where \( \alpha_\mu, \beta_\mu \) are the principal spinors. Moreover, since \( \alpha^a_\mu \beta_\nu \) and \( F + \tilde{G} \) vanish simultaneously, for the null case

\[
f_{ab} = \frac{1}{2} k_{ab} k_{ab} . \tag{3.16}
\]

where the, in general complex, \( f \) has dimensions of \( f_{\nu\mu} \) and \( k_{ab} \) is a dimensionless spinor. When \( f_{\nu\mu} \) is algebraically general, then (3.15) implies

\[
F + G = 4f_{ab} f^{ab} = -2(a_\mu \beta_\mu)^2 \neq 0. \tag{3.17}
\]

Consequently, \( f_{ab} \) can always be written as

\[
f_{ab} = \frac{1}{2} (G + i\tilde{G}) k_{ab} l_{a}, \tag{3.18}
\]

with \( l_{a} \) a second dimensionless spinor linearly independent of \( k_{ab} \) and normalized according to \( k^{\mu} l_{\mu} = 1 \) and \( l_{a} \neq 0 \). Using (3.18) in (3.17)

\[
F + G = -\frac{1}{2} (G + i\tilde{G}) \bar{G} = -i\frac{G - i\tilde{G}}{2}. \tag{3.19}
\]

Here \( \tilde{G} \) is a pseudoinvariant and without losing generality a coordinate frame can be selected such that \( \tilde{G} > 0 \), while both \( G > 0 \) and \( i\tilde{G} > 0 \). Inverting yields

\[
\tilde{G} = (F + \tilde{G} - F) / i \quad \text{and} \quad \bar{G} = (F + \tilde{G} + F) / i. \tag{3.20}
\]

All these arguments may be repeated for \( P_{\nu\mu} \). Assuming an algebraically general \( P_{\nu\mu} \) (i.e., \( P + \tilde{Q} \neq 0 \)), its spinorial image becomes

\[
P_{ab} = \frac{1}{2} (\tilde{G} + i\tilde{G}) k_{ab} l_{a} . \tag{3.21}
\]

where the non-negative (in an appropriately oriented coordinate) invariants \( \omega \) and \( \tilde{W} \) are

\[
\omega = (P + \tilde{Q} - P) / i \quad \text{and} \quad \tilde{W} = (P + \tilde{Q} + P) / i. \tag{3.22}
\]

and there exists an expression equivalent to (3.19) for \( P + \tilde{Q} \).
From (3.6), (3.18), and (3.22),
$$
\mathcal{E} + \mathcal{B} = (H_p + iH_\perp) \partial \mathcal{D} + i\partial \mathcal{H},
$$
(3.23)
and it is natural to pass from $\{P, \tilde{Q}\}$ to $\{\mathcal{E}, \mathcal{B}\}$ and from $\{P, \tilde{Q}\}$ to
$\{\mathcal{D}, \tilde{\omega}\}$ as the pairs of independent invariants. Assuming temporarily that the condition for inversions with respect to $\mathcal{D}$ and $\tilde{\omega}$,
$$
\partial (\mathcal{E}, \mathcal{B})/\partial (\mathcal{D}, \tilde{\omega}) = H_\perp^2 - H_{\mathcal{D}} H_{\tilde{\omega}} \neq 0,
$$
(3.24)
is satisfied, it is not difficult to show simultaneously the implications
$$
H_p + H_\perp = -H_\perp + iH_{\tilde{\omega}} \partial \mathcal{D} + L_{\mathcal{D}} + L_{\tilde{\omega}} = -H_\perp + iL_{\tilde{\omega}} \partial \mathcal{D} + i\mathcal{H},
$$
(3.25a)
and
$$
\{\mathcal{E} = -H_\perp, \mathcal{B} = H_{\tilde{\omega}}\} \rightarrow \{\mathcal{D} = -L_{\mathcal{D}}, \tilde{\omega} = L_{\tilde{\omega}}\}.
$$
(3.25b)
Also, a direct calculation of the energy density implies that it remains assuredly non-negative if and only if $\mathcal{E} \mathcal{D} + \mathcal{B} \tilde{\omega} > 0$; but the equality requires $H_p = 0$, which contradicts the correspondence limit, hence
$$
\mathcal{E} \mathcal{D} + \mathcal{B} \tilde{\omega} > 0,
$$
(3.26)
which from (3.25) may be equivalently expressed as
$$
H_p > 0 \text{ or } L_{\mathcal{D}} > 0.
$$
(3.27)
Similarly, a positive trace for the energy-momentum tensor
$$
\pi E_{\alpha\beta} = \left[ 1 - (\mathcal{E} \mathcal{D} + \mathcal{B} \tilde{\omega}) \right] - H \geq 0
$$
(3.28)
(a requirement of the virial theorem) is equivalent to
$$
P H_p + \tilde{Q} H_\perp - H_{\mathcal{D}} \geq 0 \text{ or } L - FL_p \tilde{Q} L_\tilde{\omega} > 0.
$$
(3.29)
We will consistently accept $P_{\alpha\beta}$ as the fundamental and $f_{\mu\nu}$ as the second object. Consequently, if there are values of $(P, \tilde{Q})$ which violate the first of either (3.27) or (3.29), they must be rejected as physically inadmissible. The conditions can be interpreted in two ways: (1) as restricting the family of admissible structure functions, (2) as restricting the physically admissible values of $(P, \tilde{Q})$. Occasionally these conditions hold for every $(P, \tilde{Q})$ as in the case of the linear theory where $H = F$. When $P_{\mu\nu}$ is null, the correspondence principle implies
$$
H_p = 1, H_\perp = 0, \text{ and } H_{\mathcal{D}} = 0
$$
(3.30)
for zero values of the invariants. Therefore,
$$
P_{\mu\nu} = f_{\mu\nu} \rightarrow P_{\alpha\beta} = f_{\alpha\beta} = \frac{1}{2} f_{\lambda\mu} x_{\lambda} x_{\mu}
$$
(3.31)
and it follows that the inequality conditions are automatically met. The inequalities (3.27) and (3.29) are important in the further development of the characteristic surface theory.

The dynamic equations of nonlinear electrodynamics presented in Sec. 2 and 3 obtain a most plausible interpretation when the concepts of the electric and magnetic field vectors and the electric and magnetic induction vectors are introduced. The content of these sections then corresponds closely to the Lorentz theory of electrons, where the structure relations
$$
D^\alpha = \epsilon E^\alpha \text{ and } \tilde{B}^\alpha = \mu \tilde{H}^\alpha
$$
(3.32)
are postulated. Due to the properties of the medium $\epsilon$ and $\mu$ may be different from unity. In our case the circumstances are comparable, but the inclusions are more general linear combinations of the intensities,
$$
D^\alpha = \frac{1}{H_p} - E^\alpha - i\frac{H_\perp}{H_p} \tilde{H}^\alpha \text{ and }
\tilde{B}^\alpha = \frac{1}{L_{\mathcal{D}}} \tilde{H}^\alpha + i\frac{L_\tilde{\omega}}{L_{\mathcal{D}}} E^\alpha.
$$
(3.33)

Also, $D^\alpha$ can be $\neq E^\alpha$, $\tilde{B}^\alpha$ can be $\neq \tilde{H}^\alpha$ due only to the basic nonlinearity (which does not require the presence of any medium). When however, for example, $H_\perp = 0$, $L_\tilde{\omega} = 0$, then $1/H_p$ plays the role of $\epsilon$ and $1/L_{\mathcal{D}}$ of $\mu$; or when the field is of null type, then automatically $D^\alpha = E^\alpha$ and $\tilde{B}^\alpha = \tilde{H}^\alpha$ due to the properties of the structure function. In addition, when the field is algebraically general, there exists a distinguished orientation determined by the energy current for which the relations among the inductions and intensities take the simple form (3.32),
$$
\epsilon = \mathcal{D} / \mathcal{D} = -2 \partial L / \partial (\mathcal{D}^2) \text{ and }
\mu = \tilde{\omega} / \tilde{\omega} = 2 \partial H / \partial (\tilde{\omega}^2).
$$
(3.34)

Consequently, there is some justification for denoting the ratios $\epsilon$ and $\mu$ as the electric permittivity and the magnetic permeability. Moreover, in the appropriate limits these equations are formally identical to the conventional equations of electrodynamics in macroscopic media with point-like sources. The complete supporting details of this interpretation of nonlinear electrodynamics are found in Refs. 1 and 20.

4. SMALL PERTURBATIONS AND CHARACTERISTIC SURFACES

In this section, the equations determining the characteristic surfaces for nonlinear electrodynamics in a gravitational field corresponding to the system's set of dynamic equations are developed. The resulting complementary partial differential equations for perturbations of various field gradients are linear with variable coefficients determined by the background fields. Therefore, in principle, they can be solved with standard techniques. The equations are manipulated in spinorial form, because the application of the notation simplifies further algebraic calculation considerably.

Our arguments are local (at a fixed point), but have analytic implications. In the theory of small perturbations, the structure equations are considered as central. Conveniently, these particular relations are algebraic, enabling their thorough investigation which eventually results in the theorems on the propagation of discontinuities in the background. The basic problem of characteristic surfaces consists of deriving the necessary conditions, from the perturbed dynamic equations, which permit the existence of nontrivial discontinuities of the first derivatives of the field variables.

A surface $S(x) = \text{const}$ (corresponding to a particular $P_{\mu\nu}$) is said to be characteristic of the object $F(x)$, if the derivatives of its small perturbations of $\delta F_{\alpha\beta}$ can possess nontrivial discontinuities on $S$. All quantities denoted by $\delta (\cdot \cdot \cdot)$ are proportional to some parameter of smallness. The discontinuities...
uous jump of \( F(x) \) at \( x \) on the surface \( S \) is defined by
\[
|\{F\}| \equiv \lim_{x \rightarrow x_{+}} |F(x_{+}) - F(x_{-})|,
\]
where \( x_{+} \) and \( x_{-} \) are points located on opposite sides of \( S \) along the normal at the point \( x \). The positive side is selected by the direction of the gradient \( n_{S} \). Outside of the surface, the background object \( F \) and its first derivatives are continuous in the entire region \( \Omega \) for which it is defined. The jump in the gradient \( F_{\sigma} \) normal to \( S \) is
\[
|\{\delta F_{\sigma}\}| = \Delta F S_{\sigma}.
\]

Infinitesimally perturbing both conditions at the boundary of \( \Omega \) and each of the various currents for a particular complete solution gives, by hypothesis, a new solution differing from the original only by small perturbations of all physically relevant variables. Application of this assumption in the usual manner to the dynamic equations provides linear equations for the small perturbations given by
\[
\begin{align*}
\text{Maxwell:} & \quad \nabla^{2} S_{\mu} + \nabla^{2} S_{\mu} A^{\mu} = 4\pi j^{A}, \\
\text{Faraday:} & \quad \nabla \cdot dS_{\mu} = 0,
\end{align*}
\]

structure
\[
\begin{align*}
\delta f_{AB} = 2H_{A}^{2} P_{AB} + 16H_{A}^{2} P_{CD} + H_{A}^{2} P_{CD} + H_{A}^{2} P_{CD} P_{AB},
\end{align*}
\]

Bianchi \( \nabla^{4} S_{\mu} - \nabla \cdot dS_{\mu} = 0 \),

Einstein \( \delta C_{ABCD} = -8H_{\mu \nu} \delta (P_{A \mu} P_{C \nu}) - 88H_{\mu \nu} P_{A \mu} P_{C \nu} + 8\pi (e + \mu) \delta (u_{A \mu} u_{C \nu}) + 8\pi (e + \mu) u_{A \mu} u_{C \nu} \),

Assuming that the perturbative jumps are continuous, but their derivatives are not (e.g., \( |\{\delta P_{\mu\nu}\}| = 0 \) but \( |\{\delta P_{\mu\nu}\}| = \Delta P_{\mu\nu} \)), define the spinor image of the surface gradient by
\[
S_{AB} = \sigma^{A}_{\mu} S_{\mu} - S_{AB}^{S_{A}} S_{B} = -\delta^{A}_{B} S^{A} S_{B}^{S_{A}},
\]

and utilizing the set (4.3), the spinor form of the discontinuity equations on the surface can be written in the basic block:
\[
\begin{align*}
\text{Maxwell:} & \quad S_{A}^{B} A^{B} F_{A} = 0, \\
\text{Faraday:} & \quad S_{A}^{B} F_{A} = 0,
\end{align*}
\]

from structure \( \delta f_{AB} = 2H_{A}^{2} P_{AB} + 16H_{A}^{2} P_{CD} + H_{A}^{2} P_{CD} P_{AB} \),

from Bianchi \( S_{A}^{B} \Delta C_{ABCD} + S_{A}^{B} C_{BCD} = 0 \),

from Einstein \( \Delta C_{ABCD} = -8H_{A}^{2} C_{A} + 4 \Delta C_{A} + 4 \Delta C_{A} \),

The remainder of this work is effectively devoted to the evaluation of the properties of the solutions of these relations for the discontinuities in the derivatives of the various field variables on the characteristic surfaces. Specifically, the necessary conditions permitting the existence of \( \Delta P_{AB} \neq 0 \) are derived for background fields which are null \( P_{\mu\nu} = 0 \) and algebraically general \( P + \tilde{Q} = 0 \). The case of a trivial background field \( P_{\mu\nu} = 0 \) is omitted, since for weak fields the theory coincides identically with the linear electrodynamic case by construction. It must be underlined that we assume \( \Delta P_{AB} \neq 0 \), but whether \( \Delta f_{AB} \), etc., vanish or not remains questionable. The possible characteristic surfaces are repeatedly divided into two classes: (1) the null characteristic surfaces (NCS), where \( S_{\mu}^{S_{A}} S_{A}^{S_{A}} = 0 \), and (2) the general characteristic surfaces (GCS), where \( S_{\mu}^{S_{A}} S_{A}^{S_{A}} \neq 0 \). The symbol \( S \) is used to denote the product \( S_{\mu}^{S_{A}} S_{A}^{S_{A}} \).

5. Characteristic Surfaces of Linear Electrodynamics with Gravitation

Reducing the basic characteristic surface equations (4.5) for \( P_{\mu\nu} = \mu_{\mu} \), which corresponds to the linear electrodynamic case, Eqs. (3.30) are encountered with \( H_{A}^{2} \) and \( H_{A}^{2} \), real and in general nonvanishing. Moreover, \( C_{A} = \tilde{C}_{A} \), with \( R = 0 \). The subcases \( \Delta f_{AB} \) zero or nonzero are evaluated separately.

A. \( \Delta f_{AB} \neq 0 \) on the characteristic surface

Substituting \( P_{AB} = f_{AB} \) into (4.5a,b) and noting (4.4), one has necessarily
\[
\text{det} S^{AB} \neq 0 \Rightarrow \text{det} S^{AB} = 0 = \Delta f_{AB} = 0 = \Delta f_{A} \Delta f_{B},
\]

where \( S_{A} \) is a specific spinor which is not difficult to calculate. Select \( \{S_{A}, S_{A}^{S_{A}}\} \) with \( S^{A} S_{A} = 1 \) as a spinor basis for decomposing \( \Delta f_{A} \), such that
\[
\Delta f_{AB} = \Delta f_{A} S_{A} S_{B} + \Delta f_{A} S_{A} S_{B} + \Delta f_{A} S_{A} S_{B},
\]

Contracting this with \( S^{B} \) account for (4.5a,b) \( \Delta f_{A} = 0 = \Delta f_{A} \), thus for \( \Delta f_{A} \) a complex function,
\[
\Delta f_{AB} = \Delta f_{A} S_{A} S_{B}.
\]

Substituting \( C_{A} \) from the introductory paragraph above and (5.1) into the Bianchi identity (4.5d) yields
\[
S^{A} \Delta C_{A} = \Delta f_{A} S^{A} S_{B} S_{B} = 0.
\]

For jumps in the totally symmetric conformal curvature, the most general form is
\[
\Delta C_{A} = \Delta C_{A} S_{A} S_{B} S_{C} S_{D} + 4 \Delta C_{A} S_{A} S_{B} S_{C} S_{D} + 6 \Delta C_{A} S_{A} S_{B} S_{C} S_{D} + 4 \Delta C_{A} S_{A} S_{B} S_{C} S_{D} + \Delta C_{A} S_{A} S_{B} S_{C} S_{D}.
\]

Comparing (5.5), therefore, \( \Delta C_{A} \) is either algebraically general or null but of the form \( f_{A} = \Delta f_{A} (S_{A} - S_{B} - S_{B} - S_{B}) \), since \( \Delta C_{A} = 0 \), the result is
\[
\Delta C_{A} = \Delta C_{A} S_{A} S_{B} S_{C} (\Delta C_{A} S_{D} + 8 \Delta f_{A} S_{D}),
\]

which is of type III:.[3-1] If and only if \( f_{A} \) is null and such that \( f_{A} S^{A} S_{B} = 0 \), then \( S_{A} \) is an eigenvector of the background null field. When this circumstance occurs \( \Delta C_{A} \) vanishes directly and \( \Delta C_{A} \) is given by the first term of (5.7), which is of type N:[4].
B. $\Delta f_{AB} = 0$ on the characteristic surface

The jump relation from the first Bianchi identity (4.5d) for this subcase requires

$$S^{DA}\Delta C_{ABCD} = 0 \quad \text{if} \quad \Sigma \neq 0$$
$$\Delta C_{ABCD} \text{ is of type } N[4] \quad \text{if} \quad \Sigma = 0.$$  \hfill (5.8)

Summarizing this straightforward special case, the characteristic surfaces of linear electrodynamics are necessarily null ($\Sigma = 0$) if there exists a nontrivial jump. Specifically, the discontinuities in the derivatives of small perturbations in both the electrodynamical and gravitational fields propagate at "the speed of light," and the only discontinuous jumps in the perturbations of conformal curvature permitted are of the types III:[3-1] and N:[4].

6. CHARACTERISTIC SURFACES OF NONLINEAR ELECTRODYNAMICS WITH GRAVITATION

Regarding $P_{AB}$ as the fundamental object providing a nonlinear electrodynamical solution completing the Einstein equations,

$$C_{ABCD} = -8H_P P_{AB} P_{CD},$$

$$-1R = \lambda + 2(ZH_Z + \dot{Z}H_Z - H),$$

and selecting $\Delta P_{AB}$ as the generator of the remaining discontinuities, the set of equations (4.5) describe the physical system. Introduce the notation

$$\Delta Z: = 8P_{AB} \Delta P_{AB}$$

and the traceless energy–momentum tensor of linear electrodynamics,

$$\tau^{\mu\nu}[P]: = -P^\mu P_\mu + _{4}\cdots P^\mu P^{\nu\rho\sigma} P_{\rho\sigma}$$

$$= 4P_{AB} P_{CD} S^{AC} S^{BD}.$$  \hfill (6.3)

Then after recognizing that

$$S_{\mu} S^{\nu} \delta^B_b = : \Sigma \delta^B_b,$$

and after substituting (4.5c) into (4.5b) using (4.5a), you deduce that successively multiplying by $S_{\mu} S^{\nu}$ and $-2P^A_B$ gives

$$[\{(H_P + ZH_Z) \Sigma + H_{ZZ} \tau^{\mu\nu} S_{\mu} S_{\nu}\} \Delta Z$$

$$+ [ZH_{ZZ} \Sigma + H_{ZZ} \tau^{\mu\nu} S_{\mu} S_{\nu}] \Delta Z = 0.$$  \hfill (6.5)

This expression and its complex conjugate form a system of linear homogeneous equations for $\Delta Z$ and $\Delta Z$, whose determinant is

$$\Delta: = \left| \begin{array}{cc}
(H_P + ZH_Z) \Sigma + H_{ZZ} \tau^{\mu\nu} S_{\mu} S_{\nu} & \\
ZH_{ZZ} \Sigma + H_{ZZ} \tau^{\mu\nu} S_{\mu} S_{\nu} & \end{array} \right|^2$$

$$- \left| \begin{array}{cc}
ZH_{ZZ} \Sigma + H_{ZZ} \tau^{\mu\nu} S_{\mu} S_{\nu} & \\
(H_P + ZH_Z) \Sigma + H_{ZZ} \tau^{\mu\nu} S_{\mu} S_{\nu} & \end{array} \right|^2,$$

which must vanish whenever $\Delta Z \neq 0$ as a direct algebraic consequence of the discontinuity relations. In general, we claim

$$\Delta P_{AB} \neq 0 \implies \Delta \neq 0.$$  \hfill (6.7)

This implication is conveniently proved by contradiction; hence assume that simultaneously $\Delta P_{AB} \neq 0$ and $\Delta \neq 0$. From (6.5), $\Delta Z = 0$ and then $H_P \Sigma \Delta P_{AB} = 0$, or for $H_P > 0$ we infer a NCS with $S_{\mu} S^{\nu} \delta^B_b = 0$. $S_{\mu} S^{\nu}$ has the algebraic form

$$S_{\mu} S^{\nu} = S_\mu S^{\nu} \text{ which, when introduced into (4.5a and b), yields the expressions}$$

$$\Delta P_{AB} S_{\mu} S^{\nu} = \Delta P_{\mu} S_{\nu} \quad \text{and} \quad \Delta f_{AB} S_{\mu} S^{\nu} = \Delta f_{\mu} S_{\nu}.$$  \hfill (6.8)

where $\Delta P$ and $\Delta f$ are both real numbers. Using the reduced form of the structure relation (4.5c) in the second of these equations necessitates

$$\Delta f = 2iH_Z \Delta P - \Delta P = \Delta f,$$  \hfill (6.9)

because the real part of $2H_Z$ is $H_P > 0$ by hypothesis. Consequently, there exist nonvanishing complexes $\Delta P'$ and $\Delta f'$ such that

$$\Delta P_{AB} = \Delta P' S_{\mu} S_{\nu} \quad \text{and} \quad \Delta f_{AB} = \Delta f' S_{\mu} S_{\nu}$$

with

$$\Delta f' = 2H_Z \Delta P'$$

which due to $\Delta Z = 8P_{AB} \Delta P_{AB} = 0$, implies

$$P_{AB} S_{\mu} S^{\nu} = 0.$$  \hfill (6.11)

Independent of whether the background is trivial ($P_{AB} = 0$) or nontrivial ($P_{AB} = P_{\mu} S_{\nu} \neq 0$), we encounter

$$\Sigma = 0 \quad \text{and} \quad \tau^{\mu\nu} S_{\mu} S_{\nu} = 0,$$

requiring that $\Delta = 0$, a contradiction. Thus the implication has been established. It remains to demonstrate the existence of nontrivial $\Delta P_{AB}$ (and $\Delta f_{AB}$) when $\Delta = 0$, which will be done by explicit calculation in what follows.

A. The class of NCS ($\Sigma = 0$)

Proceeding as in the proof above, Eqs. (4.5a)–(4.5c) lead to (6.8) with

$$\Delta f = 2iH_Z \Delta P - 2\Delta P \cdot P_{AB} + \Delta Z \cdot S_{\mu} S_{\nu} = 1,$$  \hfill (6.13)

where

$$\Delta H_Z: = H_{ZZ} \Delta Z + H_{ZZZ} \Delta Z$$

$$= 8\Re H_{ZZ} P_{\mu} P_{\nu} + H_{ZZ} P_{AB} + H_{ZZ} P_{AB}.$$  \hfill (6.14)

The determinant reduces to

$$\Delta: = \left| H_{ZZ} ^2 - H_{PP} H_{QQ} \right|^2 - \left| \tau^{\mu\nu} S_{\mu} S_{\nu} \right|^2$$

$$= \left| H_{PP} H_{QQ} - H_{PP} H_{QQ} \right|^2 - \left| 2P_{AB} S_{\mu} S^{\nu} \right|^2 = 0.$$  \hfill (6.15)

1. Equation (6.15) and $H_{QQ} \Delta Z$ is zero.

This can hold locally for specific values of $P$ and $\Omega$ or globally for all admissible values, from the structure of $H$. Consider $F: = F \cdot H_{QQ}$ whose differential is given by

$$H_{PP} F_{Hr} + H_{QQ} F_{Ht} \cdot DP + (H_{PP} F_{Ht} + H_{QQ} F_{Ht}) dQ.$$  \hfill (6.16)

Observe that it is possible to encounter a nontrivial $F$ with vanishing differential, because the linear homogeneous system of equations

$$H_{PP} F_{Hr} + H_{QQ} F_{Ht} = 0, \quad H_{PP} F_{Ht} + H_{QQ} F_{Ht} = 0$$

has vanishing determinant. This subcase may globally yield the existence of a $F(H_{PP}, H_{QQ}) = \text{const}$, which constrains the possible structure functions. The linear theory ($H = P$) is an important, and trivial, subclass belonging to the case discussed.

Satisfying (6.13) with $\Delta H_Z = 0$ implies in a manner identical to the proof of (6.7) that $\Delta P = 0 = \Delta f$, leading again to (6.10). When $P_{\mu} S_{\nu} \delta^B_b \neq 0$ also

$$H_{ZZ} ^2 - H_{ZZZ} H_{ZZ} = 0$$

if there exists a nontrivial solution, suggesting a further separation into two subcases. If $\Delta H_Z$ of

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(6.14) vanishes due to \( H_{zz} = H_{zz} = 0 \), then \( \Delta P' \) remains arbitrary and \( \Delta P_{AB} \) contains two arbitrary constants. Or, for the alternative subroutine, if \( |H_{zz}| = |H_{zz}| \neq 0 \), you encounter

\[
\frac{\Delta P'}{\Delta P'} = -\frac{H_{zz} P_{AB} S^{\beta} S^\beta}{H_{zz} P_{AB} S^{\beta} S^\beta},
\]

(6.17)

indicating the phase of the jump \( \Delta P' \) is determined when the structure function and field are known. Only \( |\Delta P'| \) remains arbitrary and \( \Delta P_{AB} \) depends on one arbitrary constant.

Substituting the jump expression from the Einstein equations (4.5e) into that of the first Bianchi identity (4.5d) and again using the content of (6.10), you obtain

\[
\Delta C_{ABCD} = S_{A} S_{B} S_{C}(\Delta C_{1} S_{D}) - 2H_{P} \Delta P_{AB} S^{\gamma} S^\gamma S_{D} B_{I} \]

(6.18)

where \( \Delta C_{1} \) is complex and arbitrary. Consequently, the jump in conformal curvature on the characteristic surface is of type III:[3-1], except when the background nonlinear electrodynamic field is null where the gradient \( S_{\mu} \) is the same vector resultant from a jump of type \( N[4] \).

Equation (6.15) and

\[
H_{P} P_{AB} S^{\gamma} S^\gamma = 0 \rightarrow P_{AB} = P(A S_{B}).
\]

When the background is this trivial the implication is automatically fulfilled; but if \( P_{AB} \neq 0, S_{\mu} \) is an eigenvector of \( P_{\mu \nu} \), i.e., the relation above may be translated to the equivalent tensorial image

\[
P_{\mu} S_{\mu} = \lambda S_{\mu} \Rightarrow \star P_{\mu} S_{\mu} = 0,
\]

(6.19)

where for algebraically general \( P_{AB} = \frac{1}{2}(\partial + i \\tilde{\partial}) S_{B}, k'_{\mu} = 1 \rightarrow \exists \) real nonvanishing \( \lambda = \{ \partial, \tilde{\partial} \} \), depending upon whether \( S_{\mu} = k_{\mu} \) or \( l_{\nu} \), and where for null \( P_{AB} = 1 \), \( \{ 4P_{kS} k_{\nu} P_{\lambda} = 0 \}

Satisfying (6.13) when \( \Delta H_{zz} = 0 \), one concludes immediately from the structure relation (4.5c) that

\[
\Delta f_{AB S_{B}} = 2H_{zz} \Delta P_{AB} S_{B} + 2P_{AB} \Delta H_{zz}.
\]

(6.20)

The background field acquires the form \( P_{AB} = P(A S_{B}) \), where (6.19) holds with

\[
\Delta f = 2i \Delta P H_{zz} = \Delta P H_{zz} P^{A S_{B}}
\]

(6.21)

and

\[
\Delta H_{zz} = 8i \Delta P (H_{zz} P^{A S_{A}} - H_{zz} P^{A S_{A}}) \neq 0,
\]

(6.22)

which implies \( \Delta P \neq 0 \). Hence the structure function must be nonlinear so that at least one of \( H_{zz} \) or \( H_{zz} \) is different from zero; and the complex invariant of the field \( Z = 4P^{A B} P_{AB} \neq 0 \), hence the field is algebraically general. Transforming to the invariants \( \mathcal{D} \) and \( \mathcal{H} \) defined by

\[
Z := p + \mathcal{Q} = -\frac{1}{2} [\mathcal{D} + i \mathcal{H}]^{2}
\]

(6.23)

and noting that, since \( \Delta f \) and \( \Delta P \neq 0 \) are real, the structure function is constrained by the condition \( H_{zz} = 0 \), we have

\[
2 \Delta f = -4 \Delta P H_{zz} H_{zz}.
\]

(6.24)

Also

\[
\Delta P_{AB} S^{\beta} A^{\gamma} + i \Delta P S^\gamma A_{\beta} \neq \Delta f_{AB} S^{\gamma} A_{\beta} + \Delta f S^{\gamma} A_{\beta}.
\]

(6.25)

with \( \Delta f = 2H_{zz} \Delta P' \), where \( \Delta f' \) and \( \Delta P' \) are in general complex. Therefore, the general forms for the jumps are

\[
\Delta P_{AB} = 2\Delta P S^{\gamma} A_{\beta} + 2i \Delta P S^\gamma A_{\beta},
\]

(6.25a)

\[
\Delta f_{AB} = 2\Delta f' S^{\gamma} A_{\beta} + 2\Delta f S^{\gamma} A_{\beta},
\]

(6.25b)

Here \( \Delta P_{AB} \) depends on three arbitrary constants due to the complex \( \Delta P' \) and the real \( \Delta P \). The background field is given by

\[
P_{AB} = \frac{1}{2} 4 \mathcal{D} + i \mathcal{H} S_{B}.
\]

(6.26)

The determinant condition \( \Delta = 0 \) is automatically satisfied and

\[
\Delta H_{zz} = i (\Delta P / 4 P^{A S_{B}}) [H_{zz} - i (H_{zz} + i H_{zz})].
\]

(6.27)

Regarding the relevance of these last electrodynamic jumps, observe that the constraint \( H_{zz} \neq 0 \) contradicts the correspondence assumption \( H_{zz} = (\mathcal{D} - \mathcal{H}) \) for small values of \( \mathcal{D} \) and \( \mathcal{H} \). Therefore, probably the only cases of physical interest are those possessing constant valued invariants for some set of points. From (6.25), \( \Delta P_{\mu

\nu} \) cannot be a null bivector; yet if additionally \( H_{zz} = 0 \), then \( \Delta f = 0 \) and \( \Delta f_{\mu \nu} \) is a null bivector (or even trivial if \( \Delta P = 0 \rightarrow \Delta f' = 0 \). If this pathologic situation occurs, then \( \mathcal{D} (\mathcal{H}, \mathcal{H}) \) does not exist for the corresponding values of \( \mathcal{D} \) and \( \mathcal{H} \).

Examining the first Bianchi identity (4.5d) for this subcase and multiplying independently by \( \mathcal{D} \) and \( \mathcal{H} \) gives

\[
S^{\gamma} \Delta C_{ABCD} = 4H_{P} [\mathcal{D} \mathcal{D} + i \mathcal{H} + i \mathcal{H}] P^{A S_{B} S_{C}} + 2 \mathcal{H} P^{A S_{B} S_{C}} - 2 \mathcal{H} P^{A S_{B} S_{C}},
\]

(6.28)

and

\[
|Z| \Delta H_{zz} + 2 \Delta H_{zz} = 0 \rightarrow \text{Re} \Delta H_{zz} = 0.
\]

(6.29)

But

\[
\Delta H_{zz} = 4 \Delta P \mathcal{D} (H_{zz} \mathcal{H} - (\mathcal{D} - \mathcal{H}) H_{zz}),
\]

(6.30)

hence, setting the real part to zero places an additional restriction on the structure function,

\[
\mathcal{H} H_{zz} + \mathcal{D} (H_{zz} \mathcal{H} + H_{zz} \mathcal{H}) = 0.
\]

(6.31)

The most general form of the curvature jump becomes

\[
C_{ABCD} = S_{A} S_{B} |\Delta C_{1} S_{C} + a S_{C} (S_{D} + b S_{D})|
\]

(6.32)

where \( a + b \Delta C_{1} = 16 H_{P} (\mathcal{D} - \mathcal{H}) P^{A} \) and \( a-b = 16 H_{P} \mathcal{H} P^{A} \). That is, \( \Delta C_{ABCD} \) is to type \( D[2-1-1] \) in this subcase.

**B. The class of GCS (\( S \neq 0 \))**

Multiplying by \( S_{\mu} A \) the result of substituting (4.5c) into (4.5b) and using (4.5a) gives

\[
H_{P} S^{\gamma} P^{A} A^{\gamma} + |H_{zz} A^{\gamma} + H_{zz} A^{\gamma} | \sigma P^{A} - |H_{zz} A^{\gamma} + H_{zz} A^{\gamma} | S_{A} S^{A} P^{A} = 0,
\]

(6.33)

Under the basic assumption of the class, solving this for the electrodynamic jump \( \Delta P_{AB} \) yields

\[
\Delta P_{AB} = S_{A} S^{A} P_{AB} \Delta w,
\]

(6.34)
\[
\Delta w = -(H_{zz} \Delta Z + H_{zz} \Delta Z |H_p \Sigma)^{-1}.
\]

Substituting (6.34) into (6.2) and the result into (6.35), we obtain
\[
\rho \Delta w + \sigma \Delta \bar{w} = 0,
\]
where
\[
\rho = (H_p + 2ZH_{zz}) \Sigma + 2H_{zz} r^{\nu} S_{\mu} S_{\nu},
\]
\[
\sigma = 2ZH_{zz} \Sigma + 2H_{zz} r^{\nu} S_{\mu} S_{\nu}.
\]
Equation (6.36) and its complex conjugate form a linear homogeneous set of equations whose determinant is precisely \(\Delta\). When \(\Delta \neq 0\), necessarily
\[
\Delta = |\rho|^2 - |\sigma|^2 = 0,
\]
which coincides with (6.6). Since \(\Delta P_{ab} \neq 0\), GCS can exist only when (1) the background field is nontrivial \(P_{ab} \neq 0\), (2) the jump \(\Delta w \neq 0\) (which occurs only when \(\Delta Z \neq 0\)), and (3) the situation is genuinely nonlinear (at least one of \(H_{zz}\) or \(H_{zt}\) must be nonzero). We acknowledge that from the structure of (6.34), \(\Delta P_{\nu} r^{\nu} S_{\mu} S_{\nu} = 0\). The same was true for NCS.

Before proceeding with the consideration of null and algebraically general background fields, it is profitable to analyze some algebraic consequences of the Bianchi identities. Multiplying both Bianchi identities (4.5d) by \(\text{SAD}\), and taking the symmetric and antisymmetric parts result in
\[
\Sigma C_{ABCD} = S(A S B C D + B C D),
\]
\[
S(A S B C D + B C D) = 0,
\]
\[
S(A S B C D + B C D) = 0.
\]
For GCS, using the expression for \(\Delta P_{ab} (6.34)\) and the Einstein equation (6.1),
\[
\Delta C_{ABC} = -8H_p (S_A S_B S_C S_D P_{CD}) \Delta w + S_A S_B S_C S_D P_{CD} \Delta w,
\]
\[
-\frac{1}{2} \Delta R = \Delta (H_p P + H Q Q - H) = \Delta H_p + \Delta H_{zz},
\]
Using \(H_{zz} = -H_p \Sigma \Delta w, H_{zz} = -H_p \Sigma (\Delta w + \Delta \bar{w}) \Sigma,\) and \(H_{Q Q} = -H_{Q Q} (\Delta w - \Delta \bar{w}) \Sigma,\) these imply
\[
S_{AB} \Delta C_{ABC} = -H_p \Sigma (Z \Delta w + Z \Delta \bar{w}) S_{AC},
\]
\[
\frac{1}{2} \Delta R = H_p \Sigma (Z \Delta w + Z \Delta \bar{w}).
\]
Substituting (6.40a) into (6.39a) provides
\[
\Sigma \Delta C_{ABCD} = -8H_p \Sigma \Delta w P_{ab} P_{CD} + \Delta w S_A S_B S_C S_D P_{CD} + \Delta w S_A S_B S_C S_D P_{CD}.
\]

1. **Null background electrodynamic field \((P_{ab} = [P k_A k_B])\)**

\(P \neq 0\) is in general complex. Due to correspondence for the partial evaluated at \(Z = 0, H_p = 1, H_{Q Q} = H_{P Q} = 0\) such that \(4H_{zz} = H_{pp} + H_{Q Q}^2, 4H_{zz} = H_{pp} - H_{Q Q}^2\). The expressions for \(\rho\) and \(\sigma\) become
\[
\rho = \Sigma + \frac{1}{2} (H_{pp} - H_{Q Q}) r^{\nu} S_{\mu} S_{\nu},
\]
\[
\sigma = \frac{1}{2} (H_{pp} + H_{Q Q}) r^{\nu} S_{\mu} S_{\nu},
\]
and
\[
\Delta = \rho^2 - \sigma^2 = (\rho + \sigma)(\rho - \sigma) = 0.
\]
There exist two alternatives, either characteristic type
\[
P: \rho + \sigma = \Sigma + H_{pp} r^{\nu} S_{\mu} S_{\nu} = 0
\]
\[
Q: \rho - \sigma = \Sigma - H_{Q Q} r^{\nu} S_{\mu} S_{\nu} = 0,
\]
with \(r^{\nu} S_{\mu} S_{\nu} \neq 0\).

When \(Z = 0\), in the exceptional case where
\[
H_{pp} + H_{Q Q} = 0 \quad \text{with} \quad \rho = 0 = \sigma,
\]
the two characteristics coincide and there exists a single characteristic equation
\[
\Sigma = H_{Q Q} r^{\nu} S_{\mu} S_{\nu} \neq 0,
\]
where the inequality is valid when the single surface is GCS. Equation (6.36) is automatically satisfied with arbitrary complex \(\Delta w \neq 0\), and hence the solution for \(\Delta P_{ab} (6.34)\) contains two real arbitrary constants. Calculating the product \(4\Delta P_{ab} \Delta P_{ab}\) demonstrates that \(\Delta P_{\mu} \nu\) is algebraically general, although it is a simple bivector. If \(S_\mu \rightarrow \lambda k_\mu,\) then the GCS limits to a NCS, the characteristic equation remains fulfilled, and \(\Delta P_{\mu} \nu\) becomes a null bivector.

In the general case where
\[
H_{pp} + H_{Q Q} \neq 0
\]
the characteristic surfaces are distinct, and one finds
\[
P: \rho = -\sigma = -\frac{1}{2} (H_{pp} + H_{Q Q}) r^{\nu} S_{\mu} S_{\nu} \neq 0,
\]
\[
Q: \rho = \sigma = \frac{1}{2} (H_{pp} + H_{Q Q}) r^{\nu} S_{\mu} S_{\nu} \neq 0
\]
when the surfaces are GCS. Correspondingly, condition
(6.36) reads, respectively,
\[ P: \Delta w + \tilde{\Delta} w = 0 \quad \text{and} \quad \tilde{Q}: \Delta w - \tilde{\Delta} w = 0. \]  
(6.50)

Thus, \( \Delta w \) is either purely real or imaginary, and \( \Delta P_{ab} \) contains only one arbitrary real (multiplicative) constant, \( \Delta P \), is algebraically general in both cases. Notice, it may be that one surface is a NCS, as in the situation where \( H = H(P) = H_{0} = 0 \); then the above argument applies only to the one GCS.

In a normalized spinor base \( \{ k_{a}, l_{a} \} \) such that \( k^{A}l_{a} = 1 \), \( S_{ab} \) may be expanded in the form
\[ S_{ab} = xk_{a}k_{b} + yl_{a}l_{b} + zk_{a}l_{b} + \lambda_{a}k_{b}. \]  
(6.51)

with \( x, y \) real and \( z \) complex, giving
\[ - \Sigma = \Sigma^{AB}S_{AB} = xy - z\bar{z} \neq 0 \]
and
\[ - \tau^{a}S_{a}S_{n} = \frac{1}{2}|xy + z\bar{z}| \neq 0. \]  
(6.52)

Respecting the constraints on the null background field dictated by correspondence and defining
\[ \eta^{A} = \eta^{B}k^{A} + \eta^{D}l^{A}, \]  
(6.53)

with \( \eta^{4}S_{4}k_{a} = -(\eta_{x} - \eta_{y}z), \) (6.42) has the form
\[ \Sigma A_{BCD} = -\frac{1}{2}[(xy - z\bar{z})^{2}P^{2}\Delta w_{k}k_{b}k_{c}k_{d} + P^{2}\Delta w_{l}l_{c}l_{b}l_{c}l_{d}] \times [y(\eta_{B} + zl_{c})(\eta_{D}l_{b} + zl_{c})]. \]  
(6.54)

\[ 2p = [H_{p} + H_{x}H_{y} - i(H_{x} + iH_{\bar{q}})]xy - [H_{p} - H_{y} + iH_{\bar{q}}]z\bar{z}, \]  
(6.59)

\[ 2\sigma = -(\mathcal{D} + i\mathcal{D}_{\bar{q}})\mathcal{D}^{-1}[(H_{p} - H_{x} + iH_{\bar{q}}) xy + (H_{p} + H_{y} - iH_{\bar{q}}) z\bar{z}], \]
which give for (6.38)
\[ \Delta = (H_{p}xy + H_{x}z\bar{z})[H_{p} - H_{y} + iH_{\bar{q}}] - (H_{p} + iH_{\bar{q}})^{2}xy\bar{z} = 0. \]  
(6.60)

The discriminant of this quadratic form in \((xy)\) and \((z\bar{z})\) is
\[ d = [H_{p} + H_{x}H_{y} - H_{p}^{2} - (H_{x} + iH_{\bar{q}})^{2} + 4H_{p}^{2}] H_{p} H_{x} H_{y} \geq 0. \]  
(6.61)

Defining
\[ \tau \mu = \frac{1}{2H_{p}} \left\{ \sqrt{d} + \left[ H_{p}^{2} + (H_{x} + iH_{\bar{q}})^{2} - H_{x}H_{y} \right] \right\}, \]
(6.62)

and supposing \( H_{x} H_{y} \neq 0 \), the determinant becomes
\[ \Delta = \frac{H_{p}}{H_{x}H_{y}} \left( H_{x}H_{y} xy - \tau z\bar{z} \right)[H_{p} - \tau z\bar{z}]. \]  
(6.63)

We claim that \( \tau \) is strictly positive when \( H_{p} > 0 \), independent of the values of the other derivatives of \( H \). To prove the assertion assume \( \tau \leq 0 \); then \( \sqrt{d} > 0 \) implies
\[ \lambda := H_{p}^{2} + (H_{x} + iH_{\bar{q}})^{2} - H_{x}H_{y} \leq 0. \]  
(6.64)

The definition of \( d \) and \( \tau \) necessitates \( \sqrt{d} < -\lambda \rightarrow d = \lambda^{2} \). Since \( H_{p} \neq 0 \) this requires
\[ H_{p}^{2} + (H_{x} + iH_{\bar{q}})^{2} - \lambda < 0 \rightarrow H_{p} = 0, \]  
(6.65)
which is a contradiction of the assumption \( H_p > 0 \). Therefore \( \tau > 0 \) and the determinant can be factored such that
\[
\Delta = (H_p/\tau)(H_{\bar{x} \bar{y}} xy - r\bar{z}z)(\tau xy + H_{\bar{y}y} \bar{z}z).
\] (6.66)

Although it was originally assumed that \( H_{\bar{x} \bar{y}} \neq 0 \), the result remains valid when \( H_{\bar{x} \bar{y}} \rightarrow 0 \) because, in the final result, this derivative has disappeared from the denominators. When the background field is algebraically general, the two equations for the characteristic surfaces, obtained from \( \Delta = 0 \), are
\[
\text{I: } \tau xy + H_{\bar{y}y} \bar{z}z = 0 \quad \text{and} \quad \text{II: } H_{\bar{x} \bar{y}} xy - r\bar{z}z = 0.
\] (6.67)

These equations are linearly dependent if and only if \( d = 0 \). Using (6.58), these may be rewritten into the two alternative characteristic types,
\[
\text{I: } (\tau - H_{\bar{x} \bar{y}}) \Sigma + (1/|Z|)(H_{\bar{x} \bar{y}} + \tau)\nu^{\mu}S_{\nu}S_{\nu} = 0,
\]
\[
\text{II: } (H_{\bar{x} \bar{y}} + \tau) \Sigma + (1/|Z|)(H_{\bar{x} \bar{y}} - \tau)\nu^{\mu}S_{\nu}S_{\nu} = 0.
\] (6.68)

As a consistency criteria, (6.68) for null background fields must reduce to (6.45) by the limiting transition \( Z = P + O \rightarrow -O \) (i.e., \( \partial \rightarrow i\bar{\partial} \rightarrow -O \)). Applying the correspondence condition in the limit, \( \tau \) becomes unity and the coefficients of the characteristic equations conform to the conclusion
\[
\text{if } H_{pp} + H_{\bar{x} \bar{y}} > 0, \text{ I } \rightarrow \text{P and II } \rightarrow \bar{Q},
\] (6.69)
\[
\text{if } H_{pp} + H_{\bar{x} \bar{y}} < 0, \text{ I } \rightarrow \bar{Q} \text{ and II } \rightarrow P.
\]
The linearly dependent case, \( d = 0 \), occurs if and only if \( H_{\bar{x} \bar{y}} + iH_{\bar{y}y} = 0 = H_{\bar{x} \bar{y}} + H_{\bar{y}y} \). (6.70)

Under these conditions \( \tau = H_{pp} \), reducing both characteristic equations for the single surface to
\[
(H_{\bar{x} \bar{y}} xy - H_{\bar{y}y} \bar{z}z = 0 \rightarrow H_{pp} \bar{z}z = 0 \rightarrow \bar{H}_{pp} x\bar{z}z = 0)
\] (6.71)
or
\[
(H_{\bar{x} \bar{y}} + H_{pp}) \Sigma + (1/|Z|)(H_{\bar{x} \bar{y}} - H_{pp})\nu^{\mu}S_{\nu}S_{\nu} = 0.
\] (6.72)

From some straightforward algebraic manipulation of (6.70), (6.71), (6.59), and (6.61), one deduces
\[
d = 0 \rightarrow \rho = 0 = \sigma.
\] (6.73)

Therefore, with \( d = 0 \), (6.36) holds indenitely for any \( \Delta w \neq 0 \), and \( \Delta P_{AB} \) from (6.34) contains two arbitrary constants (the real and imaginary parts of \( \Delta w \)). If we assume
\[
\Delta P_{AB} = 4\Delta P_{a\bar{b}} \Delta P_{\bar{a}b},
\]
we again determine that this invariant cannot vanish and remain consistent with the characteristic equation (6.71). Consequently, \( \Delta P_{ab} \) is an algebraically general (but simple) bivector. Notice that a positive sign for \( H_{\bar{x} \bar{y}} \) is not unexpected for a reasonable structure function since correspondence requires it for sufficiently small invariants.

Assume now \( d \neq 0 \); then \( \Delta \) factors into two linear forms which are not proportional, giving two distinct characteristic equations of the form (6.67) where the coefficients are all real. Suppose that the integral of at least one equation, say (6.73) and (6.38), \( d \neq 0 \rightarrow |\rho| = |\sigma| \neq 0 \), and \( \Delta w \) has the form
\[
\Delta w = \Delta \lambda (2i \rho + |\sigma|),
\] (6.74)
with \( \Delta \lambda \) real, arbitrary, and nonzero. From (6.34), \( \Delta P_{AB} \) contains only one arbitrary parameter \( \Delta \lambda \) fixing the phase. Analogously to the previous cases, for GCS, e.g., \( d = 0 \), the expression \( \Delta P_{AB} \Delta P_{\bar{a}b} \) is nonvanishing, so that \( \Delta P_{ab} \) is an algebraically general (but simple) bivector.

When the background field is algebraically general \( (P_{AB} = -4(\partial + i\bar{\partial})(\zeta_{AB} l_b), \) (6.42) can be rewritten in the form
\[
\Delta C_{ABCD} = 4H_p \Sigma^{-1}\bar{Z}\Delta \bar{w} [x^2 \zeta_{AB} k_A k_C k_D + y^2 \zeta_{AB} l_b l_c l_d + 2(xy + z\bar{z})\zeta_{AB} l_b l_c l_d + 2(xy + z\bar{z})\zeta_{AB} l_b l_c l_d]
\] (6.75)
If the characteristics satisfy the second of (6.67) or are of type II, the invariants are
\[
z = (H_{\bar{x} \bar{y}} / \tau)(xy)^{1/2} e^\lambda, \quad \Sigma = (H_{\bar{x} \bar{y}} / \tau - 1)xy.
\] (6.76)

Defining \( H_{\bar{x} \bar{y}} / \tau = \xi \), the conformal curvature jump is
\[
\Delta C_{ABCD} = 4\Delta P_{a\bar{b}} \Delta P_{\bar{a}b} / (\xi - 1)^{-1}\bar{Z} \Delta \bar{w} [\xi (x^2 \zeta_{AB} k_A k_C k_D + y^2 e^{-2i\lambda} l_b l_c l_d) + 2(1 + \xi) \zeta_{AB} \zeta_{AB} l_b l_c l_d + 2(xy + z\bar{z})\zeta_{AB} l_b l_c l_d]
\] (6.77)
which has the distinct roots
\[
\eta \pm = \xi^{-1/2} (1 + \xi) \pm (\xi - 1)(\bar{Z} \Delta \bar{w} / \bar{Z} \Delta \bar{w} + 2\xi(\xi - 1)^{-1/2})^{1/2}
\] (6.78)

Therefore, \( \Delta C_{ABCD} \) is of type I: \( |1 - 1 - 1 - 1| \).


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Similarly, if the characteristics are of type I,

\[ x = (-|H_{\mu\nu}/\tau \xi\zeta|)^{1/2}, \quad y = (-|H_{\mu\nu}/\tau \xi\zeta|)^{1/2} e^{-\alpha}, \quad \Sigma = (1 + H_{\mu\nu}/\tau)z\zeta, \]  

(6.80)

where \( \alpha \) is real. Defining

\[ \tilde{k}_\xi = e^{\omega/2} \tilde{k}_\xi, \quad \tilde{I}_\zeta = e^{-\omega/2} \tilde{I}_\zeta, \quad \tilde{A}_\zeta = (z\zeta)^{1/2}; \]

one obtains the same result for the jump except for an overall change of sign and an interchange of \(\alpha\) and \(\beta\).

For GCS, each jump expression for the conformal curvature [specifically (6.42) and its consequences (6.54) and (6.75)] becomes to conformally flat when the perturbation in the electrodynamic field vanishes. When this circumstance occurs or, that is, when \(\Delta w = 0\), (6.42) requires

\[ \Sigma \Delta C_{ABCD} = 0 \implies \Delta C_{ABCD} = 0 \text{ if } \Sigma \neq 0 \]  

(6.81)

Therefore, whenever there exists a nonvanishing discontinuous jump in the perturbation of the conformal curvature with \(\Delta w = 0\), the jump is of type N:4 if and the characteristic surfaces have degenerated to NCS. Certainly, for this subcase the number of free parameters in the jump also returns to the values appropriate for the corresponding NCS.

### 7. Metric Properties of General Characteristic Surfaces

For general characteristic surfaces (\(\Sigma \neq 0\)), the conditions distinguishing the surfaces \(S(x) = 0\) as timelike with spacelike gradient (\(\Sigma > 0\)) or spacelike with timelike gradient (\(\Sigma < 0\)) are now investigated. Under the first circumstance the characteristic cones are interpreted as being inside of the standard cone, and in the second as outside. Certain structure function properties can permit the propagation of discontinuities of the derivatives in excess of the "speed of light". For convenience, we introduce the following notation: \(CS \equiv \) characteristic surface, \(T \equiv \) timelike, \(S \equiv \) spacelike, and \(N \equiv \) lightlike or null.

#### A. Analyzing GCS with null background electrodynamic field

\[ P_{\mu\nu} = \frac{1}{2} P_{\xi\zeta} k_\xi k_\zeta \rightarrow \tau_{\mu\nu} = |P|^{-2} k_\mu k_\nu, \quad k_\mu k_\mu = 0, \quad \text{we infer from (6.45) that} \]

\[ P: \quad \Sigma = -H_{\mu\nu}|P|^{-2} k_\mu S_\nu, \quad \hat{Q}: \quad \Sigma = -H_{\mu\nu}|P|^{-2} k_\mu S_\nu. \]  

(7.1)

Consequently,

\[ \begin{align*}
\text{CS of type } P & \text{ is } & T & \text{ if } H_{\mu\nu} < 0, \\
\text{CS of type } P & \text{ is } & N & \quad H_{\mu\nu} = 0, \\
\text{CS of type } P & \text{ is } & S & \quad H_{\mu\nu} > 0, \quad \text{if } H_{\mu\nu} > 0
\end{align*} \]  

(7.2a)

\[ \begin{align*}
\text{CS of type } \hat{Q} & \text{ is } & T & \quad H_{\mu\nu} > 0, \\
\text{CS of type } \hat{Q} & \text{ is } & N & \quad H_{\mu\nu} = 0, \\
\text{CS of type } \hat{Q} & \text{ is } & S & \quad H_{\mu\nu} < 0. 
\end{align*} \]  

(7.2b)

B. For GCS with an algebraically general background field where \(\Sigma = xy - z\zeta\), types I and II of (6.67) are decomposed separately

Introducing the definitions

\[ X_i = H_{\mu\nu}^2 - H_{\mu\nu} H_{\mu\nu}, \]

\[ Y_i = (H_{\mu\nu} - H_{\mu\nu})(H_{\mu\nu} + H_{\mu\nu}) - (H_{\mu\nu} + iH_{\mu\nu})^2 = 4|Z|^2 X, \]  

(7.3)

\( X \neq 0 \) assures that neither of (6.67) can reduce to \(xy - z\zeta = 0\), or null surfaces can occur only in the exceptional situation where simultaneously \(xy = 0 = z\zeta\).

For characteristics of type I from (6.67),

\[ \tau xy + H_{\mu\nu} z\zeta = 0 \implies \Sigma = (H_{\mu\nu}/\tau + 1)z\zeta, \]  

(7.4a)

CS is \(S \equiv -H_{\mu\nu}/\tau - 1 > 0 \implies -H_{\mu\nu} > 0 \) and \(\tau < -H_{\mu\nu}\) with

\[ (I) \quad Y > 0 \Rightarrow X > 0 \text{ for } Z \neq 0 \text{ and/or} \]

\[ (II) \quad Y > H_{\mu\nu} (H_{\mu\nu} + H_{\mu\nu}) > 0. \]  

(7.5a)

CS is \(N \equiv -H_{\mu\nu} \equiv \tau. \)

For characteristics of type II from (6.67),

\[ H_{\mu\nu} xy - \tau z\zeta \equiv \Sigma = -\left(\tau/H_{\mu\nu} - 1\right)z\zeta, \]

(7.6)

where

\[ CS \equiv \tau = H_{\mu\nu} > 1 < 0 \quad \text{if } H_{\mu\nu} < 0. \]  

(7.7a)

CS with \(I)\) \(H_{\mu\nu} < 0\) is \(T. \)

\[ (II) H_{\mu\nu} > 0 \text{ is (a) } T \equiv \tau < H_{\mu\nu} \equiv Y \]

\[ + H_{\mu\nu} H_{\mu\nu} > 0 \Rightarrow Y > 0 \]

\[ \Rightarrow X > 0 \text{ for } Z \neq 0. \]  

(7.7b)

CS is \(S \equiv \tau = H_{\mu\nu} < 0 \), which follows if

\[ (I) Y + H_{\mu\nu} H_{\mu\nu} > 0 \text{ or} \]

\[ (II) Y + H_{\mu\nu} H_{\mu\nu} > 0 \text{ and } Y < 0. \]

(7.7c)

CS is \(N \equiv \tau = H_{\mu\nu}. \)

In synthesis we deduce

\[ \begin{align*}
\text{I and II are } & \text{ and } H_{\mu\nu} > 0, \quad -H_{\mu\nu} > 0, \\
\Rightarrow & H_{\mu\nu} H_{\mu\nu} > 0 > Y > 0, \\
\text{I is } & \text{ and II is } T, \\
H_{\mu\nu} > 0 & \Rightarrow \left| \begin{array}{c}
\text{I is } T \text{ and II is } S, \\
-H_{\mu\nu} > 0 & \Rightarrow H_{\mu\nu} > 0 \text{ and } -Y > 0 \end{array} \right|, \\
Y > H_{\mu\nu} H_{\mu\nu} & \Rightarrow |, \\
\text{I is } & \text{ T and II is } S, \\
H_{\mu\nu} > 0 & \Rightarrow |> 0. \\
\end{align*} \]  

(7.8a)

(7.8b)

(7.8c)
The conditions for the four alternatives (7.8) are mutually exclusive; therefore, they are not only necessary but also sufficient and the implication \( \Rightarrow \) may be replaced by \( \Leftrightarrow \) in each. Notice, finally, that discontinuities of the derivatives of small perturbations of the nonlinear electrodynamic field are propagated within the light cone if and only if (7.8) is satisfied, which is equivalent to

\[
(H_p/2(|H_{pp} - H_{QQ}|) > H_{QQ}^2 - H_{pp} H_{QQ} > 0. \quad (7.9)
\]

In the case where \( d = 0 \) the surfaces coincide, and the resultant conditions may be read from (7.5) and (7.7) by recalling that \( \tau = H_p \) for this situation. There are of course other special cases, for example, a complete set where one characteristic surface is null and (6.15) necessitates that \( X = 0 \). But a detailed investigation of these generally pathological cases would be excessive pedantry, since a sufficient number of relevant cases has already been displayed.

Next we reconsider the eiconal equation \( \gamma^{\mu\nu} S^{\mu}_{s} S^{\nu}_{s} = 0 \) for GCS, where nonlinear electrodynamic waves are transported along bicharacteristic rays which are null with respect to the metric \( \gamma^{\mu\nu} \) as opposed to \( g^{\mu\nu} \). Directly from the characteristic surface equations (6.45), for the case of a null electromagnetic field, the eiconal equation implies

\[
P: \gamma^{\mu\nu} \sigma_{\mu
u} = \phi^{-2} (g^{\mu
u} + H_{pp} \tau^{\mu
u}),
\]

\[
\tilde{Q}: \gamma^{\mu\nu} \sigma_{\mu
u} = \phi^{-2} (g^{\mu
u} - H_{QQ} \tau^{\mu
u}),
\]

where \( \phi \) is an arbitrary conformal factor. As a note, these metrics coincide if and only if the structure function is the Hamiltonian of Born–Infeld NLE. \(^2\) Also from (6.68), for the case of an algebraically general electromagnetic field, the eiconal equations imply

\[
\gamma^{\mu\nu} = \phi^{-2} \left[ g^{\mu\nu} + \frac{1}{|Z|} (H_{\underline{p}\underline{p}} + \tau)/(H_{\underline{p}\underline{p}} + \tau) \mu\nu \right],
\]

(7.11)

**CONCLUSION**

The dynamic equations for a charged perfect fluid with nonlinear electromagnetic interaction in a gravitational space–time are deduced from the fundamental action. The characteristic surfaces for this physical system are found to have discontinuous first derivatives of small perturbations in the fields (1) which may be of general types and (2) which

---

**TABLE I. Types of discontinuous jumps.**

<table>
<thead>
<tr>
<th>Characteristic equations</th>
<th>Restrictions on ( H )</th>
<th>( \Delta P_{\mu\nu} )</th>
<th>Limit</th>
<th>No. of parameters in ( \Delta P_{\mu\nu} )</th>
<th>( \Delta C_{\mu\nu} )</th>
<th>No. of parameters in ( \Delta C_{\mu\nu} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Sigma = 0 )</td>
<td>( P_{\mu\nu} S_{\mu} S_{\nu} \neq \lambda S_{\mu} )</td>
<td>( \Delta H_{i} = 0 )</td>
<td>[2]</td>
<td>( \Delta P \rightarrow 0 )</td>
<td>[3-1] - [4]</td>
<td>4 - 2</td>
</tr>
<tr>
<td>( \Sigma = 0 )</td>
<td>( P_{\mu\nu} S_{\mu} S_{\nu} = \lambda S_{\mu} H_{p} ) ( \neq 0 )</td>
<td>( \left</td>
<td>H_{zz} \right</td>
<td>= \left</td>
<td>H_{zz} \right</td>
<td>\neq 0 )</td>
</tr>
<tr>
<td>( \Sigma = 0 )</td>
<td>( P_{\mu\nu} S_{\mu} S_{\nu} \neq \lambda S_{\mu} ) ( H_{p} ) ( \neq 0 )</td>
<td>( \left</td>
<td>H_{zz} \right</td>
<td>\neq 0 )</td>
<td>[1-1]</td>
<td>( \Delta P \rightarrow 0 )</td>
</tr>
<tr>
<td>( \Sigma = 0 )</td>
<td>( P_{\mu\nu} S_{\mu} S_{\nu} \neq \lambda S_{\mu} ) ( H_{p} ) ( \neq 0 )</td>
<td>( \left</td>
<td>H_{zz} \right</td>
<td>\neq 0 )</td>
<td>[1-1]</td>
<td>( \Delta P \rightarrow 0 )</td>
</tr>
<tr>
<td>( \Sigma = 0 )</td>
<td>( P_{\mu\nu} S_{\mu} S_{\nu} \neq \lambda S_{\mu} ) ( H_{p} ) ( \neq 0 )</td>
<td>( \left</td>
<td>H_{zz} \right</td>
<td>\neq 0 )</td>
<td>[1-1]</td>
<td>( \Delta P \rightarrow 0 )</td>
</tr>
</tbody>
</table>

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could have spacelike propagation. These perhaps unexpected properties are entirely dependent upon the specific nature of the structure function of the nonlinearity. Presumably, the introduction of other nonlinear interactions (e.g., fluid dynamic) could be anticipated to provide similar consequences. The demonstration of a physically relevant source for a nonlinear electrodynamic interaction providing causal spacelike signal propagation remains open. However, space- 

like quantum processes present possible candidates. Permitting for curved gravitational space–times and for characteristic surfaces under various restrictions, the discontinuous jumps generated are cataloged in Table I. In each case, the procedure for limiting to a vanishing electromagnetic jump is also presented. A speculative comment on the emergence of conformal jumps of type I: [1-1-1-1] is perhaps appropriate. The nonvanishing trace of the energy–momentum tensor [since \( -4R = \lambda + 2(ZH Z + \dot{Z}H Z - H) \)] implies the existence of some finite fundamental mass and, consequently, length which ought to force the breaking of the conformal group previously enjoyed by the electromagnetic equations in the linear case. Under these circumstances, jumps of general types are not implausible.

Whenever the characteristic surfaces are of types \( P \) and \( \tilde{Q} \), the discontinuities in the derivatives of small perturbations of the nonlinear electrodynamic field are propagated interior to the light cone if and only if \( H_{11} < 0 < H_{22} \).

For characteristic surfaces of types I and II, the discontinuities are propagated within the light cone if and only if

\[
(H_{11}/2)Z ||(H_{22} - H_{22}) > H_{11}^2 - H_{11}H_{22} > 0.
\]

Consequently, if the physical environment is sufficiently nonlinear, there exists a distinct chance that there are relevant structure functions permitting spacelike propagation of jumps of general types.

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15. J. Schwinger, Phys. Rev. 82, 914 (1951); 74, 1439 (1949); 75, 651 (1949); 76, 790 (1949).
16. The dual is defined for a tensor of rank \( n \) by

\[
\tilde{T}^{\mu_1 \cdots \mu_n} = \frac{1}{2^n} \frac{\epsilon^{\mu_1 \cdots \mu_n \nu_1 \cdots \nu_n}}{2^n} T^\nu_1 \cdots T^\nu_n,
\]

such that \( \tilde{T}^\nu \cdots \nu = T^\nu_1 \cdots \nu_n \).
17. A pseudoinvariant, under \( C_n \), transformations, transforms like a scalar times the sign \( \frac{\partial f}{\partial x} \frac{\partial f}{\partial x} \).
18. The conventions of J. F. Plebański, "Spinors, Tetrads, and Forms," a monograph in two volumes of the Centro de Investigación y de Estudios Avanzados del IPN, México (1975) are generally observed. Index, differentiation, and symmetrization symbols are conventional. The sign: \( (\text{equal from definition}) \) is distinguished from \( (\text{equal due to logic}) \).
19. The curvature tensor

\[
R^\nu_{\rho \gamma} = \{-\tilde{g}^\nu_{\rho} \tilde{g}^\mu_{\gamma} + \tilde{g}^\nu_{\gamma} \tilde{g}^\mu_{\rho} - \tilde{g}^\nu_{\rho} \tilde{g}^\mu_{\gamma} \}
\]

induces the Ricci tensor and scalar curvature

\[
R_{\mu \gamma} = R^\nu_{\mu \gamma} \tilde{g}^\nu_{\nu}.
\]

The Einstein and conformal curvature tensors are defined, respectively, by

\[
G_{\mu \gamma} = R_{\mu \gamma} - \frac{1}{2} \tilde{g}_{\mu \gamma} R
\]

and

\[
R_{\mu \gamma} = -C^\nu_{\mu \nu} + \tilde{g}^\nu_{\rho} \tilde{g}^\mu_{\gamma} (R_{\rho \gamma} - \tilde{g}_{\rho \gamma} R) + \tilde{g}^\nu_{\nu} R.
\]

The relation between the metric and the matrices is

\[
E_{\mu \nu} = -\tilde{g}^\nu_{\rho} E_{\mu \rho} + \tilde{g}^\nu_{\nu} E_{\mu \nu},
\]

and the spin tensor is defined by

\[
S_{\mu \nu} = \frac{1}{2} \delta_{\mu \nu} E_{\rho \nu} - \tilde{g}^\rho_{\rho} S_{\mu \nu}.
\]

The appearance of \( [\cdot] \) rather than \( [\cdot] \) and the seemingly unusual number of parameters in \( \Delta C_{\mu \nu} \), for the GCS results from \( (6.81) \).