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Real Einstein spaces constructed via linear superposition of complex gravitational fields: the concrete case of non-twisting type N solutions

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Abstract. By means of a linear superposition of a couple of tetrads, determining a self-dual type N flat space and an anti-self-dual flat type N field, the most general non-twisting type N real vacuum solution of the Einstein equation is constructed.

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1. Introduction

In spite of the big achievements of complex relativity theory [1–4], its contribution in the determination of real solutions is rather modest, due to the sophisticated procedure [5] to be used to accomplish a 'real cut' of a given complex solution of the (complex) Einstein equations.

In [6] 'one sided' $G \times [-]$, $D \times [-]$ and $N \times [-]$ complex solutions of the complex Einstein equations were provided. The authors expressed at that time the hope that such solutions could be important 'as basic and elementary bricks which, through a procedure of synthesis (at present [1976] unknown) would generate physical real solutions'.

Besides the Rozga procedure, there exists a simple mechanism based on a linear superposition of the basis 1-forms, corresponding to one sided self-dual and anti-self-dual gravitational fields, which permits one to construct real solutions from complex ones. This approach is useful, at least, for certain families of solutions. For instance, accomplishing a linear superposition of the tetrads of the $D \times [-]$ and $[-] \times D$ fields, from (5.2) of [6], one obtains the real $D \times D$ Plebanski–Demianski gravitational field.

In this brief report we are concerned with a linear superposition of self-dual and anti-self-dual non-twisting type $N$ complex spaces to build up a real solution of the Einstein equations.

We recall that working within the null tetrad formalism, the metric is given by

$$g = 2e^1 \otimes e^2 + 2e^3 \otimes e^4$$  \hspace{1cm} (1)

and the connections 1-forms $\Gamma_{ab} = \Gamma_{[ab]}$ are determined from the first Cartan structure equations

$$de^a = e^b \wedge \Gamma_{ab}.$$  \hspace{1cm} (2)

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The second structure Cartan equations, within which the vacuum Einstein equations are in-built, can be decoupled into a self-dual and an anti-self-dual subspace of the space of 2-forms. The self-dual sector is given by

\[ A := \text{d}(\Gamma_{42} + \Gamma_{42} \wedge (\Gamma_{12} + \Gamma_{34}) = \frac{1}{4} C^{(5)} S^{11} + \frac{1}{2} C^{(4)} S^{12} + \frac{1}{4} C^{(3)} S^{22} \]

\[ B := \text{d}(\Gamma_{12} + \Gamma_{34}) + 2\Gamma_{42} \wedge \Gamma_{31} = \frac{1}{2} C^{(5)} S^{11} + C^{(3)} S^{12} + \frac{1}{2} C^{(2)} S^{22} \]

\[ C := \text{d}(\Gamma_{34} + (\Gamma_{12} + \Gamma_{34}) \wedge \Gamma_{31} = \frac{1}{4} C^{(5)} S^{11} + \frac{1}{2} C^{(4)} S^{12} + \frac{1}{4} C^{(3)} S^{22} \]  \hspace{1cm} (3)

where \( S^{AB} (A, B = 1, 2) \) are the self-dual basis 2-forms.

\[ S = 2e^{A} \wedge e^{B} \]

\[ S^{11} = e^{1} \wedge e^{2} + e^{3} \wedge e^{4} \]

\[ S^{22} = 2e^{3} \wedge e^{4} \]

\[ * S^{AB} = S^{AB} \]  \hspace{1cm} (4)

The asterisk \( * \) denotes the Hodge star operation: for any \( p \)-form \( \omega \) one defines a \( p' \)-form \( *\omega \), with \( p' = 4 - p \), by

\[ * \omega = \frac{1}{p!} \frac{1}{p'}! \exp[\frac{1}{2} i \pi (pp' - 2)] e^{\alpha_{1}...\alpha_{p\prime}} \omega_{\alpha_{1}...\alpha_{p}} e^{\beta_{1}...\beta_{p'}} \wedge ... \wedge e^{\beta_{p'}}. \]

The coefficients \( C^{(a)} \), \( a = 1, \ldots, 5 \), are the Weyl complex coefficients representing the Weyl conformal tensor. The anti-self-dual sector, coexisting with the self-dual one, is given by

\[ \overline{A} := \text{d}(\Gamma_{41} + \Gamma_{41} \wedge (\Gamma_{34} - \Gamma_{12}) = \frac{1}{4} \overline{C}^{(5)} \overline{S}^{11} + \frac{1}{2} \overline{C}^{(4)} \overline{S}^{12} + \frac{1}{4} \overline{C}^{(3)} \overline{S}^{22} \]

\[ \overline{B} := \text{d}(\Gamma_{34} - \Gamma_{12}) + 2\Gamma_{41} \wedge \Gamma_{32} = \frac{1}{2} \overline{C}^{(5)} \overline{S}^{11} + \overline{C}^{(3)} \overline{S}^{12} + \frac{1}{2} \overline{C}^{(2)} \overline{S}^{22} \]

\[ \overline{C} := \text{d}(\Gamma_{32} + (\Gamma_{34} - \Gamma_{12}) \wedge \Gamma_{32} = \frac{1}{4} \overline{C}^{(5)} \overline{S}^{11} + \frac{1}{2} \overline{C}^{(3)} \overline{S}^{12} + \frac{1}{4} \overline{C}^{(1)} \overline{S}^{22} \]  \hspace{1cm} (5)

where the anti-self-dual basis 2-forms \( \overline{S}^{AB} \) are

\[ \overline{S}^{11} = 2e^{A} \wedge e^{B} \]

\[ \overline{S}^{12} = -e^{A} \wedge e^{B} \]

\[ \overline{S}^{22} = 2e^{A} \wedge e^{B} \]  \hspace{1cm} (6)

For real Einstein spaces one requires:

\[ \text{c.c.} \, e^{1} = \overline{e}^{1} \]

\[ \text{c.c.} \, e^{2} = \overline{e}^{2} \]

\[ \text{c.c.} \, e^{3} = \overline{e}^{3} \]

\[ \text{c.c.} \, e^{4} = \overline{e}^{4} \]  \hspace{1cm} (7)

where c.c. stands for complex conjugation. In complex relativity one does not impose the above relations on the 1-forms \( e^{A} \), the 2-forms \( S^{AB} \) and \( \overline{S}^{AB} \), and the Weyl curvature coefficients \( C^{(a)} \) and \( \overline{C}^{(a)} \).

From the point of view of complex relativity one may build up spaces of the form:

\[ \text{any Petrov-type space} \oplus \text{flat space} \]

by requiring the vanishing of the Weyl curvature coefficients \( C^{(a)} \), describing the curvature of the anti-self-dual subspace. Of course, one can proceed in the converse direction, namely

\[ \text{flat space} \oplus \text{any Petrov-type space} \]

There arises an obvious question: can one construct real solutions of the Einstein equations via a simple procedure, say a linear superposition of the complex one-sided solutions?

The answer is positive at least for certain classes of one-sided solutions, as we shall see from a concrete example.
2. Superposition of one-sided Petrov type $N$ fields

We shall show that a linear superposition, at the level of tetrads, of Petrov type $N$ one-sided fields, $N \otimes [0]$ and $[0] \otimes N$, produces a real solution of the Einstein equations, i.e. a solution $N \otimes N$ from the point of view of complex relativity, which allows a real cut.

Let us consider a gravitational field given in the complex chart $\{\xi, \bar{\xi}, r, t\}$ by the tetrad

$$e^1(s) = d(r\xi) + \beta dt + r[-2f + \xi f,\xi] dt$$
$$e^2(s) = d(r\bar{\xi}) + \beta dt + r\bar{\xi} f,\bar{\xi} dt$$
$$e^3(s) = -d(r\xi \bar{\xi}) + [\alpha + \xi r(2f - \xi f,\xi)] dt$$
$$e^4(s) = dr + [\bar{\epsilon} + r f,\xi] dt$$

(8)

where $\alpha$ and $\epsilon$ are real constants, $\beta$ is a complex parameter and $f(\xi, t)$ is an arbitrary function of $\xi$ and $t$ with non-vanishing third derivative $f,\xi\xi\xi \neq 0$. The italic letter $s$ is used in $e^s(s)$ to denote the self-dual character of the considered field. The Ricci connections associated with the tetrad above (8) are

$$\Gamma_{42} = 2[f - \xi f,\xi + \frac{1}{2}\xi^2 f,\xi\xi] dt$$
$$\Gamma_{12} + \Gamma_{34} = 2(f,\xi - \xi f,\bar{\xi}) dt$$
$$\Gamma_{51} = f,\xi \bar{\xi} dt$$

(9)

while

$$\Gamma_{41} = 0 \quad \Gamma_{34} - \Gamma_{12} = 0 \quad \Gamma_{32} = 0.$$  

(10)

Hence, from the second Cartan equations one establishes that the considered field is a $N \otimes [0]$ space. The Weyl curvature coefficients $C^{(a)}$, corresponding to the type $N$ self-dual field, are determined by

$$C^{(a)} = 2(-\bar{\xi})^{(a-1)}C \quad C := -\frac{1}{\psi r} f,\xi\xi\xi \neq 0 \quad a = 1, 2, \ldots, 5$$

$$\psi := \alpha + \beta \xi + \bar{\beta} \bar{\xi} + \epsilon \xi \bar{\xi}.$$  

(11)

Of course, the curvature corresponding to the anti-self-dual sector is zero, as becomes apparent from equations (5) when considering (10). Thus, $C^{(a)} = 0$.

On the other hand, one determines an anti-self-dual $[0] \otimes N$ type $N$ gravitational field, in a coordinate chart $\{\xi, \bar{\xi}, r, t\}$ by the tetrad

$$e^1(\bar{s}) = d(r\bar{\xi}) + \bar{\beta} dt + r\bar{\xi} f,\bar{\xi} dt$$
$$e^2(\bar{s}) = d(r\bar{\xi}) + \beta dt + r[-2\bar{f} + \bar{\xi} f,\xi] dt$$
$$e^3(\bar{s}) = -d(r\xi \bar{\xi}) + [\alpha + \xi r(2\bar{f} - \bar{\xi} f,\xi)] dt$$
$$e^4(\bar{s}) = dr + [-\epsilon + r f,\xi] dt$$

(12)
here \( \alpha \) and \( \epsilon \) are real constants, \( \beta \) is a complex parameter, \( \bar{f}(\bar{\xi}, t) \) is an arbitrary function of \( \bar{\xi} \) and \( t \) such that \( \bar{f}_{\xi \xi} \). The symbol \( \bar{s} \) is used in \( e^a(\bar{s}) \) to denote that we are dealing with anti-self-dual gravitational fields. In general there do not exist any relationships between the parameters and functions defining self- and anti-self-dual gravitational fields. Thus, at this level, the function \( \bar{f}(\bar{\xi}, t) \) is not the complex conjugate of the function \( f(\xi, t) \).

The non-vanishing connection 1-forms resulting from the tetrad (12) are

\[
\Gamma_{41} = 2(\bar{f} - \bar{\xi} \bar{f}_{\bar{\xi}} + \frac{1}{2} \bar{\xi}^2 \bar{f}_{\bar{\xi} \bar{\xi}}) \, dt \\
\Gamma_{34} - \Gamma_{12} = 2(\bar{f}_{\bar{\xi}} - \bar{\xi} \bar{f}_{\bar{\xi} \bar{\xi}}) \, dt \\
\Gamma_{31} = \bar{f}_{\bar{\xi} \bar{\xi}} \, dt.
\]

Substituting these expressions into the equations (5) one arrives at the curvature Weyl coefficients \( \bar{C}^{(a)} \),

\[
\bar{C}^{(a)} = 2(-\bar{\xi})^{(a-1)} \bar{C} \quad \bar{C} := -\frac{1}{r \psi} \bar{f}_{\bar{\xi} \bar{\xi} \bar{\xi}} \quad a = 1, \ldots, 5
\]

\[
\psi = \alpha + \beta \xi + \bar{\beta} \bar{\xi} + \epsilon \xi \bar{\xi}.
\]

Since the remaining \( \Gamma \)'s are equal to zero,

\[
\Gamma_{42} = 0 = \Gamma_{31} \quad \Gamma_{12} + \Gamma_{34} = 0
\]

the curvature of the self-dual sector vanishes, i.e. \( C^{(a)} = 0 \). Thus, the gravitational field defined by the tetrad (12) is of the type \( [0] \otimes N \).

3. Linear superposition procedure

In order to determine a real solution of the Einstein equation, we propose to accomplish a linear superposition of tetrads of the form

\[
e^a = \frac{1}{2}[e^a(s) + e^a(\bar{s})].
\]

Requiring the fulfilment of the complex conjugation conditions \( e^2 = c.c. e^1, e^3 = c.c. e^3, e^4 = c.c. e^4 \), the tetrad \( e^a \) determines the real solution we are looking for.

According to (16), the 'generated' tetrad for the sought \( N \otimes N \) gravitational field is given by

\[
e^1 = d(r \xi) + \bar{\beta} \, dt + r[-f + \frac{1}{2} \xi (f_{\xi} + \bar{f}_{\bar{\xi}})] \, dt \\
e^2 = d(r \bar{\xi}) + \beta \, dt + r[-\bar{f} + \frac{1}{2} \bar{\xi} (f_{\bar{\xi}} + \bar{f}_{\bar{\xi}})] \, dt \\
e^3 = -d(r \bar{\xi}) + \{\alpha + r[\bar{\xi} f + \xi \bar{f} - \frac{1}{2} \xi \bar{\xi} (f_{\xi} + \bar{f}_{\bar{\xi}})]\} \, dt \\
e^4 = dr + [-\epsilon + \frac{1}{2} r (f_{\xi} - \bar{f}_{\bar{\xi}})] \, dt
\]
where now $\xi$ and $\bar{\xi}$ are complex conjugate variables, $r$ and $t$ are real coordinates, $f(\xi, t)$ and $\bar{f}(\bar{\xi}, t)$ are complex conjugate functions, $\epsilon$ and $\alpha$ are real parameters, and $\beta$ is a complex constant. The Ricci connections, associated with the tetrad (17), are equal to

$$\Gamma_{42} = [f - \xi f_{,\xi} + \frac{1}{2} \xi^2 f_{,\xi\xi}] \, dt \quad \Gamma_{34} + \Gamma_{12} = (f_{,\xi} - \xi f_{,\bar{\xi}}) \, dt$$

$$\Gamma_{31} = \frac{1}{2} f_{,\xi\xi} \, dt \quad \Gamma_{41} = [-\bar{f} - \bar{\xi} \bar{f}_{,\bar{\xi}} + \frac{1}{2} \bar{\xi}^2 \bar{f}_{,\bar{\xi}\bar{\xi}}] \, dt$$

$$\Gamma_{34} - \Gamma_{12} = (\bar{f}_{,\xi} - \xi \bar{f}_{,\bar{\xi}}) \, dt \quad \Gamma_{32} = \frac{1}{2} \bar{f}_{,\xi\xi} \, dt. \quad (18)$$

Finally, as one might expect, the Weyl curvature coefficients are

$$C^{(a)} = (-\xi)^{(a-1)} C \quad \bar{C}^{(a)} = (-\bar{\xi})^{(a-1)} \bar{C}$$

$$C = -\frac{1}{r \psi} f_{,\xi\xi} \quad \bar{C} = -\frac{1}{r \psi} \bar{f}_{,\bar{\xi}\bar{\xi}} \quad \bar{C}^{(a)} = c.c. \, C^{(a)} \quad a = 1, \ldots, 5 \quad (19)$$

$$\psi = \alpha + \beta \xi + \bar{\beta} \bar{\xi} + \epsilon \xi \bar{\xi}.$$  

For these curvatures the Debever–Penrose equation possesses a quadrupole root. Thus, as it should be, the field obtained is of type $N$.

The metric structure, given by tetrad (17) and characterized by the Ricci connections (18) and the Weyl curvature coefficients (19), describes the most general real non-twisting type $N$ solution of the Einstein equations [7]. In fact, to arrive at the standard description (see formula (8) of [7]) of these type $N$ fields, which is given by the tetrad

$$e^1 = r \, d\xi + [\bar{\beta} + \epsilon \xi - rf] \, dt \quad e^2 = c.c. \, e^1$$

$$e^2 = r \, d\bar{\xi} + [\beta + \bar{\epsilon} \bar{\xi} - \bar{r} \bar{f}] \, dt$$

$$e^3 = \psi \, dt \quad \psi = \alpha + \beta \xi + \bar{\beta} \bar{\xi} + \epsilon \xi \bar{\xi}$$

$$e^4 = dr + [-\epsilon + \frac{1}{2} r (f_{,\xi} + \bar{f}_{,\bar{\xi}})] \, dt \quad (20)$$

one has to accomplish a tetrad $\rho$-gauge of the form

$$e^1 = e^1 - \rho e^4 \quad e^2 = e^2 - \bar{\rho} e^4 \quad e^3 = e^3 + \rho e^1 + \rho e^2 - \rho \bar{\rho} e^4 \quad e^4 = e^4 \quad (21)$$

with

$$\rho = \xi. \quad (22)$$

The tetrads appearing in the right-hand side of this transformation are assumed to be the original (generated) ones, while the primed tetrads correspond to those of equation (20).

Under a $\rho$-gauge, the Ricci connections transform as

$$\Gamma_{42}' = \Gamma_{42} + \rho (\Gamma_{12} + \Gamma_{34}) + \rho^2 \Gamma_{31} - d\rho \quad \Gamma_{41} = c.c. \, \Gamma_{42}$$

$$\Gamma_{12}' + \Gamma_{34}' = \Gamma_{12} + \Gamma_{34} + 2 \rho \Gamma_{31} \quad \Gamma_{34} - \Gamma_{12} = c.c. \, (\Gamma_{34} + \Gamma_{12}) \quad (23)$$

$$\Gamma_{31}' = \Gamma_{31} \quad \Gamma_{32} = c.c. \, \Gamma_{31}.$$
Therefore, as it should be, the connections associated with the primed tetrad (20) are
\[
\begin{align*}
\Gamma_{42'} &= -d\xi + f dt \\
\Gamma_{41'} &= -d\tilde{\xi} + \tilde{f} dt \\
\Gamma_{12'} + \Gamma_{34'} &= f_{,t} dt \\
- \Gamma_{12'} + \Gamma_{34'} &= \tilde{f}_{,\tilde{t}} dt \\
\Gamma_{31'} &= \frac{1}{2} f_{,\xi\xi} dt \\
\Gamma_{32'} &= \frac{1}{2} \tilde{f}_{,\tilde{\xi}\tilde{\xi}} dt.
\end{align*}
\]

The only non-vanishing curvature Weyl coefficient is \( C^{(1)} \),
\[
C^{(1)} = C = \frac{1}{r^2} f_{,\xi\xi\xi} \quad \overline{C}^{(1)} = \text{c.c. } C^{(1)}.
\]

Hence the direction \( e^{3'} \) is a quadrupole Debever–Penrose direction of the Weyl tensor, which reconfirms that the real generated gravitational field is of type \( N \).

4. Concluding remarks

The exhibited superposition procedure can be extended to construct other families of real spaces starting from one-sided gravitational fields. For instance, by using this procedure one can derive the well known real vacuum Kinnersley type D solutions. It would be of interest to use this superposition process to construct real solutions starting from general one-sided solutions (even of the special Petrov types) of the complex Einstein equations \([8,9]\); for instance, from a twisting type \( N \) field \( \otimes \) flat, or from complex \( N \otimes N \) or \( D \otimes D \) types.

References