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Notes on a cross product of vectors

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The relation between quaternionlike algebras and cross products of vectors is demonstrated. A classification of all cross products of vectors is given.

I. INTRODUCTION

During the preparation of our paper on quaternionlike algebras, Professor José Adem suggested that those considerations seemed to be closely related to the problem of a definition of a cross product of vectors in a vector space of an arbitrary (finite) dimension. The purpose of the present paper is to find this relation.

There are several possibilities to generalize the usual cross product of vectors in three-dimensional real Euclidean vector space $\mathbb{R}^3$ to arbitrary finite-dimensional vector spaces. Thus according to Eckmann, Whitehead, and Zvengrowski, one can define a vector cross product in an $n$-dimensional real Euclidean vector space $\mathbb{R}^n$ to be a mapping

$$ P_r: R^n \rightarrow R^n, \quad 1 \leq r < n, $$

satisfying the following axioms.

1. $P_r$ is a continuous mapping of $R^n$ ($1 \leq r < n$) into $R^n$.
2. $P_r(v_1,\ldots,v_r) \cdot v_i = 0,$ for every set of vectors $(v_1,\ldots,v_r) \in R^n$ and $i = 1,\ldots,r$.
3. $P_r(v_1,\ldots,v_r) \cdot P_r(v_1,\ldots,v_r) = \det(v_1,\ldots,v_r)$ for every set of vectors $(v_1,\ldots,v_r) \in R^n$.

Here a dot stands for the usual Euclidean scalar product of vectors in $R^n$.

Another, more "algebraic" definition of the cross product of vectors can be extracted from the works given by Brown and Gray, Gray, and Dündarer, Gürsey, and Tze (see also Refs. 6 and 7). According to these works we have the following definition: Let $V$ be an $n$-dimensional vector space over a field $F$ of the characteristic $\neq 2$ and let $(\cdot,\cdot)$ be a nondegenerate, bilinear, symmetric form on $V$. A vector cross product in $V$ is a mapping

$$ P_r: V^{\times r} \rightarrow V, \quad 1 \leq r < n, $$

satisfying the following axioms.

1. $P_r$ is an $r$-linear mapping of $V^{\times r}$ ($1 \leq r < n$) into $V$.
2. $(P_r(v_1,\ldots,v_r),v_i) = 0$ for every set of vectors $(v_1,\ldots,v_r) \in V^{\times r}$ and $i = 1,\ldots,r$.
3. $(P_r(v_1,\ldots,v_r), P_r(v_1,\ldots,v_r)) = \det(v_1,\ldots,v_r)$ for every set of vectors $(v_1,\ldots,v_r) \in V^{\times r}$.

Theorem 1: The vector cross product satisfying the axioms (1)-(3) or (2)-(c) exists if and only if $(1) \: r = 1$ and $n$ even; $(2) \: r = n - 1$ and $n$ arbitrary; $(3) \: r = 2$ and $n = 3,7$; $(4) \: r = 3$ and $n = 4,8$.

Explicit formulas for the vector cross products are also known. The cases of $r = 2$ and $n = 3,8$ or $r = 1$ and $n = 8$ are, perhaps, the most interesting ones. Thus the vector cross product in $\mathbb{R}^3$ with $r = 2$ is the usual cross product. The vector cross product in $\mathbb{R}^8$ with $r = 3$ can be defined in an elegant way in terms of octonions. Moreover, Dündarer et al. were able to give a compact, unified, covariant, and explicit formulation of various "generalized vector cross products" in $\mathbb{R}^8$ (compare also with Kleinfeld). Then the cross product of vectors in $\mathbb{R}^7$ with $r = 2$ can be defined in terms of pure octonions.

Now it appears that in the case $r = 2$ the axioms (a2)-(c2) for $P_2$ in $\mathbb{R}^n$ are equivalent to the following axioms. (a3) $P_2$ is a bilinear and skew-symmetric mapping of $\mathbb{R}^n \times \mathbb{R}^n$ into $\mathbb{R}^n$.

(b3) $P_2(v_1,v_2) \cdot v_1 = P_2(v_1,v_2) \cdot v_2 = 0,$

for any $(v_1,v_2) \in \mathbb{R}^n \times \mathbb{R}^n$.

(c3) $P_2(v_1,v_2) = (v_1 \cdot v_2) - (v_2 \cdot v_1),$ for any $v_1,v_2 \in \mathbb{R}^n \times \mathbb{R}^n$.

In the present paper we consider the vector cross product satisfying the conditions that generalize in a natural manner the axioms (a3)-(c3).

II. CROSS PRODUCT OF VECTORS

Let $V$ be an $n$-dimensional vector space over the number field $F (= R$ or $C)$. Then a mapping $\cdot \times \cdot: V \times V \rightarrow V$ is said to be a cross product in $V$ if there exists a bilinear form on $V$, $(\cdot,\cdot): V \times V \rightarrow F$ such that the following conditions hold.

(a) The mapping $\cdot \times \cdot: V \times V \rightarrow V$ is bilinear and skew-symmetric.
(b) $(v \times w) |(w) = 0,$ for any $v,w \in V$.
(c) $u \cdot (v \times w) = (u \cdot w) - (u \cdot v),$ for any $u,v,w \in V$.

A bilinear form on $V$, $(\cdot,\cdot): V \times V \rightarrow F$, for which (b) and (c) hold true, is called a bilinear form associated with...
the cross product \(\times\). If \(\times\) is the usual cross product in \(\mathbb{R}^3\) then the properties (a), (b), and (c) are satisfied with \((\cdot,\cdot)\) as the usual scalar product in \(\mathbb{R}^3\).

First we prove a simple Proposition.

**Proposition 1:** The conditions (a) and (b) are equivalent to (a) and (b'), where (b') reads:

\[
(b') (v \times w)u = (u \times v)w,
\]

for any \(u,v,w \in V\). 

**Proof:** Utilizing (a) and (b') to \((u + w) \times v = v \times (u + w)\) one finds (b'). Thus (a), (b) \(\Rightarrow\) (a), (b'). The implication (a), (b') \(\Rightarrow\) (a), (b) is evident. The proof is completed. 

Employing Proposition 1 we get easily that (a), (b'), and (c) yield the following analog of the conditions (c_1) or (c_2):

\[
(v \times w)u = (v(uw) - (wv)u)^2;
\]

for \(u,v,w \in V\). (1)

Now we prove the following proposition.

**Proposition 2:** Let \(V\) be an \(n\)-dimensional vector space over \(F\) and let \(f\) be a bilinear skew-symmetric mapping of \(V \times V \to F\) such that

\[
f(u,f(v,w)) = g(u,v)w - g(u,w)v,
\]

for any \(u,v,w \in V\), (2)

then \(g\) is a symmetric form and, for \(n > 1\), \(g\) is uniquely defined by the mapping \(f\).

**Proof:** If \(n = 1\), then every bilinear form on \(v\) is symmetric. Consider \(n > 1\). Let \(e_1, \ldots, e_n\) be a basis of \(V\). We set

\[
g(e_i,e_j) = g_{ij} \in F
\]

and

\[
f(e_i,e_j) = \sum_k C_{ij}^k e_k,
\]

\[
F \exists C_{ij}^k = - C_{ji}^k,
\]

where small Latin indices are assumed to run through \(1, \ldots, n\). By applying Eq. (2) to the basic vectors \(e_i\) with the use of (3) and (4) one gets

\[
\sum_m C_{im}^l C_{mk}^j = g_{li} \delta_j^k - g_{kj} \delta_i^l,
\]

where \(\delta_i^j\) is the Kronecker delta. Contracting both sides of the formula (5) with respect to the indices \(l, k\), we obtain

\[
\sum_m C_{im}^l C_{mj}^k = - (n - 1)g_{ij}.
\]

Since \(n > 1\),

\[
g_{ij} = g_{ji} = - \frac{1}{n - 1} \sum_m C_{im}^l C_{mj}^k.
\]

Thus the proof is completed. 

As a consequence of Proposition 2 one has immediately the following Corollary.

**Corollary 1:** If a mapping \(\times : V \times V \to V\) is a cross product in \(V\), then a bilinear form on \(V\), \((\cdot,\cdot) : V \times V \to F\), fulfilling the condition (c), is symmetric and if \(\dim V > 1\), then \((\cdot,\cdot)\) is uniquely defined by \(\times\).

If \(\dim V = 1\), then (evidently) any bilinear form on \(V\) is symmetric and it is associated with the cross product \(V\) which is now uniquely defined, i.e., \(v \times w = 0\) for any \(v,w \in V\). If \(\dim V > 1\), then a bilinear form associated with a cross product in \(V\) is symmetric and it is uniquely defined by the given cross product.

Let us recall the definition of a quaternion-like algebra\(^1\) (qI algebra). An \((n + 1)\)-dimensional algebra \((n > 1)\) \(Q\) with unity \(e_0\) over \(F\) is said to be a quaternion-like algebra (qI algebra) if it is associated and if there exists a decomposition

\[
Q = F e_0 \oplus V,
\]

where \(V\) is an \(n\)-dimensional vector subspace of \(Q\) such that for every vector \(v \in V\), \(v \in V e_0\).

As it has been shown in Ref. 1, if some algebra, associative or not, admits a decomposition of the form (8), then this decomposition is unique. We need the notion of equivalent cross products. Let \(u, v, w\) be \(n\)-dimensional vector spaces over \(F\) and let \(f_1, f_2, f_3\) be cross products in \(V_1, V_2, V_3\) respectively. Then the cross product \(f_2\) is said to be equivalent to the cross product \(f_3\) if there exists an isomorphism \(i : V_1 \to V_2\) such that

\[
f_3(i(u), i(v)) = f_2(u, v),\]

for any \(u, v \in V_1\). (9)

Now we can prove the main theorem of this paper.

**Theorem 2:** Given an \((n + 1)\)-dimensional qI algebra \(Q\) over \(F\) that decomposes according to (8), we define two mappings \(g, f : V \times V \to F\) and \(f : V \times V \to F\) as follows:

\[
uvw = - g(u,v)e_0 + f(v,w),\]

for any \(u, v, w \in V\). (10)

Then \(f\) is a cross product in \(V\) and \(g\) is a bilinear form associated with \(f\).

Conversely, given an \(n\)-dimensional \((n + 1)\) vector space \(V\) over \(F\), a cross product in \(V\), \(\times : V \times V \to V\), and a bilinear form \((\cdot,\cdot)\), \(V \times V \to F\) associated with \(\times\), there exists a unique (with the precision to an isomorphism) \((n + 1)\)-dimensional qI algebra \(Q\) over \(F\) that decomposes according to (8) admitting an isomorphism \(i : V \to V\) such that

\[
f(i(u), i(v)) = (i \times i)(u, v),\]

for any \(i, u, v \in V\), (11)

i.e., \(\times\) is equivalent to \(f\), and

\[
g(i(u), i(v)) = (i \times i)(u, v),\]

for any \(i, u, v \in V\), (12)

where the mappings \(f, g \times V \to V\) and \(g, f : V \times V \to F\) are defined by (10); moreover if \(\dim V = n > 1\), then \(Q\) is uniquely (i.e., with the precision to an isomorphism) defined by the pair \((V, \times)\).

**Proof:** Let \(Q\) be an \((n + 1)\)-dimensional qI algebra over \(F\) that decomposes according to (8), and let \(g, f : V \times V \to F\) and \(f : V \times V \to F\) be the mappings defined by (10). From the fact that \(Q\) is an qI algebra it follows that \(g\) is a bilinear form on \(V\) and \(f\) is a bilinear mapping of \(V \times V \to V\). Since \(Q\) is a qI algebra, \(v \in V e_0\) for every \(v \in V\). Hence, by (10), \(f(v, v) = 0\) for every \(v \in V\). Consequently, as \(f : V \times V \to V\) is a bilinear mapping, it is skew symmetric.

Since \(Q\) is an associative algebra,

\[
(u, v)w = u(vw),\]

for any \(u, v, w \in V\). (13)

From (12) and (10) it follows that

\[
(f(u, v)w) = (f(u, v)w),\]

for any \(u, v, w \in V\). (14)

\[
f(f(u, v)w) = f(f(u, v)w) = g(u, v)w - g(v, w)u,\]

for any \(u, v, w \in V\). (15)

Let \(e_1, \ldots, e_n\) be a basis of \(V\) and let \(g, f, e_0\) be defined by (3) and (4); small Latin indices are assumed to run through \(1, \ldots, n\). The formula (15), when applied to the basic vectors \(e_i\), yields

Contracting both sides of (16) with respect to the indices \( l, k \), and then with respect to \( l, i \), one finds
\[
-\sum_m (C_{lm} C_{mk} + C_{km} C_{mj}) = g_{ij} \delta^l_k - g_{jk} \delta^l_i.
\]  
(16)

Writing (18) for \( k = i \) and adding the results to (17), we get
\[
(n+1)(g_{ij} - g_{ji}) = 0.
\]
Hence \( g_{ij} = g_{ji} \). Thus we arrive at the conclusion that the bilinear form \( g \) is symmetric. Employing the cyclic sum with respect to \( u, v, w \) for both sides of (15) one obtains the "Jacobi identity,"
\[
f(f(u,v),w) + f(f(w,u),v) + f(f(v,w),u) = 0,
\]
for any \( u, v, w \in V \).
\[
(19)
\]
Finally, from (15) and (19) one finds
\[
f(v,f(w,u)) = g(u,v)w - g(v,w)u, \quad \text{for any } u, v, w \in V.
\]
\[
(20)
\]
For the condition (14) for \( u = v \) yields
\[
g(v,f(u,w)) = 0, \quad \text{for any } u, v, w \in V.
\]
\[
(21)
\]
Comparing (20) and (21) with (c) and (b), and also employing the symmetry condition of \( g \), we conclude that the mapping \( f: V \times V \to V \) is a cross product in \( V \); moreover, let the mapping \( f: V \times V \to F \) be a bilinear form associated with \( f \). Thus the first part of our theorem has been proved.

Now let \( V \) be an \( n \)-dimensional (\( n \geq 1 \)) vector space over \( F \) and let the mapping \( \cdot \times \cdot : V \times V \to V \) be a cross product in \( V \); moreover, let the mapping \( (\cdot, \cdot) : V \times V \to F \) be a bilinear form associated with \( \cdot \times \cdot \). Let us define \( Q = F \oplus V \). We have \( Q = F e_0 \oplus V \), where \( e_0 = (1,0) \in F \oplus V \) and \( V \) is the vector subspace of \( F \oplus V \) consisting of the vectors of the form \((0, \tilde{v})\), where \( \tilde{v} \in \tilde{V} \). Define a multiplication on \( Q \) as follows:
\[
Q \times Q \ni ((a, \tilde{v}), (b, \tilde{w})) \mapsto (a, \tilde{v}) (b, \tilde{w}) \in Q,
\]
where
\[
(a, \tilde{v}) (b, \tilde{w}) := (ab - \tilde{v} \cdot \tilde{w}, a \tilde{w} + b \tilde{v} + \tilde{v} \times \tilde{w})
\]
(22) [compare (22) with Refs. 2 and 6]. It is a straightforward matter to show that \( Q \) with the above defined multiplication (22) constitutes an \((n + 1)\)-dimensional ql algebra over \( F \). Let \( i: \tilde{V} \to V \) be the natural isomorphism of \( \tilde{V} \) onto \( V \) defined as follows:
\[
i: \tilde{V} \ni \tilde{v} \mapsto (0, \tilde{v}) \in V.
\]
(23)
Then, from (22) and (23) we obtain
\[
(0, \tilde{v}) (0, \tilde{w}) = (- (\tilde{v} \times \tilde{w}), (0, \tilde{w}) (1, 0) + (0, \tilde{v}) \times \tilde{w})
\]
\[
= (- (\tilde{v} \times \tilde{w}), 0, 0 + i (\tilde{v} \times \tilde{w}), \quad \text{for any } \tilde{v}, \tilde{w} \in \tilde{V}.
\]
(24)
Comparing with (10) one gets (11) and (12).

Now let \( Q_1 = F e_0 (1) \oplus V \) be a ql algebra such that there exists an isomorphism \( i_1: \tilde{V} \to V \), for which the analogs of (11) and (12) hold. Define an isomorphism \( i_0: F e_0(1) \oplus V \to F e_0(1) \oplus V \) as in (10) for every \( a \in F \). Then it easy to check that the mapping \( i_0 \circ i^{-1}_1 \) is an isomorphism of the ql algebra \( Q_1 \) onto the ql algebra \( Q \).

Finally, utilizing Corollary 1 we complete the proof. \( \blacksquare \)

The main consequence of Theorem 2 is that there exists a 1:1 correspondence between the class of all nonequivalent vector cross products in vector spaces of dimension \( n > 1 \) and the class of all nonisomorphic ql algebras of dimension \( n + 1 > 2 \). Therefore, employing the results of our previous paper 3 concerning the classification of ql algebras, we arrive at the following conclusion.

A cross product in an \( n \)-dimensional real vector space \( V \times \cdots \times V \to V \) belongs to one of the following types:

(I) a trivial cross product, i.e.,
\[
u \times w = 0, \quad \text{for any } u, v, w \in V.
\]
(25)

(II) a nilpotent cross product of the nilpotency class 2, i.e., a nontrivial cross product such that
\[
u \times (v \times w) = 0, \quad \text{for any } u, v, w \in V.
\]
(26)

(III) there exists a basis \( e_1, \ldots, e_n \) of \( V \) such that
\[
ed_1 \times e_2 = 0, \quad e_2 \times e_3 = e_1, \quad e_3 \times e_4 = e_2, \quad e_4 \times e_5 = e_3,
\]
(27)

(IV) \( n = 3 \), the usual cross product,
\[
e_1 \times e_2 = e_3, \quad e_2 \times e_3 = e_1, \quad e_3 \times e_1 = e_2.
\]
(28)

for some basis \( e_1, e_2, e_3 \); (IV) \( n = 3 \) and there exists a basis \( e_1, e_2, e_3 \) such that
\[
e_1 \times e_2 = - e_3, \quad e_2 \times e_1 = e_3, \quad e_3 \times e_2 = e_1.
\]
(29)

Employing Proposition 2 one can easily find the bilinear forms associated with the above listed cross products.

(1) For \( n = 1 \) there exists a nonzero vector \( e_1 \) such that \( (e_1 | e_1) = \pm 1 \), or one has
\[
|u|v)w = 0, \quad \text{for any } u, v, w \in V.
\]
(31)

For \( n > 1 \), Eq. (31) holds.

(II) The formula (31) holds true.

(III) \( (e_1 | e_2) = (e_2 | e_1) = 0, \quad (e_1 | e_3) = - 1, \quad (e_2 | e_3) = 1, \quad (e_3 | e_1) = 0, \quad (e_3 | e_2) = 1, \quad (e_1 | e_2) = 1, \quad (e_2 | e_1) = 1, \quad (e_1 | e_3) = - 1, \quad (e_2 | e_3) = 1, \quad (e_3 | e_2) = - 1, \quad (e_3 | e_1) = 1.
\]

In the case of complex \( V \) we have the types (I), (II), (III), and (IV) \( n = 1 \) there exists a nonzero vector \( e_1 \) such that \( (e_1 | e_1) = 1, \) or one has (31).

Remark: Given a cross product \( \cdot \times \cdot : \tilde{V} \times \tilde{V} \to \tilde{V} \), the algebra \( (\tilde{V} \times \cdots \times \tilde{V}) \) appears to be a Lie algebra isomorphic to the Lie algebra \( \mathfrak{L}(V, \cdot, \cdot) \) induced by \( Q = F \oplus V \) with multiplication defined by (22). An isomorphism is given as follows: \( \tilde{V} \ni \tilde{v} \mapsto (0, \tilde{v}) \in V \). (For the notation see Theorem 2.) Regarding induced Lie algebras, see Ref. 1.)
Concluding the present paper we would like to deal with two problems.

The first one concerns the possibility of generalization of the cross product of vectors on the vector spaces over an arbitrary field $F$. Employing the results of our recent work one can easily realize that such a generalization does exist for an arbitrary field $F$ of the characteristic "not 2" and, in fact, it is almost "automatic." In particular, the main theorem of the present paper, i.e., Theorem 2, holds true in that general case. However, the proof of this theorem given here must be slightly changed to be valid generally. Namely, the formula (7) should be replaced by the formula

$$g_{ij} = g_{ji} = - \sum_k C_{ik}^m C_{jk}^m, \quad k \neq j, \quad (7')$$

which follows from (5) or (20). [Thus the formula (20) yields the symmetry of the bilinear form $g$ and we do not need the formulas (16)-(18).]

The canonical forms of all possible vector cross products can be written down in the case when a field $F$ is of the characteristic "not 2" and, if $n \neq 3$, also "not a divisor of $n - 3"$ (for details, see Ref. 15).

The second problem we would like to deal with concerns a relation of our cross product of vectors to Clifford algebras. This problem in all its details has been considered in Ref. 15. Now we cite the main results obtained. Given an $(n + 1)$-dimensional $(n \geq 1)$ quaternionlike algebra $Q$ over a field $F$ of the characteristic "not 2" which decomposes according to the formula

$$Q = Fe_0 \oplus Fe_1 \oplus Fe_2 \oplus Fe_3 = 1,0, e_1, e_2, e_3 = - e_2;$$

where the set of vectors $(e_1, e_2, e_3)$ constitutes a basis for $V$. Consequently we conclude that Clifford algebras over a field $F$ of characteristic $\neq 2$ define vector cross products according to the "natural" scheme given in Theorem 2 if and only if their dimension = 2 or 4.

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